

Stability and instability of standing waves for a generalized Choquard equation with potential

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Abstract

We are going to study the standing waves for a generalized Choquard equation with potential:

$$-i\partial_t u - \Delta u + V(x)u = (|x|^{-\mu} * |u|^p)|u|^{p-2}u, \quad \text{in } \mathbb{R} \times \mathbb{R}^3,$$

where $V(x)$ is a real function, $0 < \mu < 3$, $2 - \mu/3 < p < 6 - \mu$ and $*$ stands for convolution. Under suitable assumptions on the potential and appropriate frequency ω , the stability and instability of the standing waves $u = e^{i\omega t}\varphi(x)$ are investigated.

Keywords: Stability; Instability; standing wave; ground state solution; generalized Choquard equation.

1 Introduction and main results

In this paper, we are going to study the following nonlocal Schrödinger equation :

$$\begin{cases} -i\partial_t u - \Delta u + V(x)u = (|x|^{-\mu} * |u|^p)|u|^{p-2}u, & \text{in } \mathbb{R} \times \mathbb{R}^3, \\ u(0, x) = u_0(x), \end{cases} \quad (1.1) \quad \boxed{\text{ME}}$$

where $V(x)$ is a real valued function, $0 < \mu < 3$ and $2 - \mu/3 < p < 6 - \mu$. This equation arises in physics as an effective description of a non-relativistic bosonic system with two-body interactions in its mean field limit, it is also known to describe the propagation of electromagnetic waves in plasmas [1] and plays an important role in the theory of Bose-Einstein condensation [6]. This equation, which is also called the Hartree equations or the Schrödinger-Newton equations, has attracted a great deal of attention in theoretical over the past years.

As we all know, the Cauchy problem of nonlinear generalized Choquard equation has been intensively studied since the pioneering work by Chadam and Glassey in [7]. We refer the readers to [3, 14, 16] for a complete overview of the literature on the topic

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of the Cauchy problem and asymptotic behavior of the solutions. In this paper we are interested in the standing wave type solutions, i.e. solutions of the form

$$u(x) = e^{i\omega t}\varphi(x), \quad (1.2) \quad \boxed{\text{SWS}}$$

where $\omega > 0, \varphi \in H^1(\mathbb{R}^3) \setminus \{0\}$ satisfies the following nonlocal elliptic equation:

$$-\Delta\varphi + \omega\varphi + V(x)\varphi - (|x|^{-\mu} * |\varphi|^p)|\varphi|^{p-2}\varphi = 0. \quad (1.3) \quad \boxed{\text{SE}}$$

For the nonlocal Schrödinger equations with $V(x) \equiv 0$, Chen and Guo in [8] studied

$$\begin{cases} i\varphi_t + \Delta\varphi + \left(\frac{1}{|x|^\alpha} * |\varphi|^p\right)|\varphi|^{p-2}\varphi = 0, \\ \varphi(x, 0) = \varphi_0(x), \quad x \in \mathbb{R}^3. \end{cases}$$

and proved the instability of the standing wave solution. For the Cauchy problem of the Hartree equation with harmonic potential, that is $V(x) = |x|^2$, we refer the readers to book [3] and the references therein. In [9], the authors derived a variant of Gagliardo-Nirenberg interpolation inequality involving nonlocal nonlinearity and determined its best (smallest) constant. The authors also established a sharp criterion for the global existence and blow-up of solutions of the Hartree equation with harmonic potential. In [24] the author obtained the blow-up and strong instability result via construction of a cross-constrained invariant set. While in [5], the authors studied the ground states of

$$-\Delta u + \omega u = (|x|^{-1} * |u|^2)u, \quad \text{in } \mathbb{R}^3.$$

and considered the stability of the standing waves for a class of Hartree equation. In [28], the classical limit of (1.1) with harmonic potential and $p = 2$ was studied by Carles, Mauser and Stimming. We would like to mention that a recent paper [27] where the authors investigated the soliton dynamics for the Hartree equation by proving stability estimates in the spirit of Weinstein for local equations. subsequently, without using the uniqueness and nondegeneracy of the ground states of the generalized Choquard equation, the authors in [26] also studied the soliton dynamics behavior. For the stability and instability of standing wave of nonlinear local Schrödinger equation, we may refer the readers to [2, 4, 11, 12, 22, 23, 30, 31].

The aim of this paper is to consider the instability and stability of standing waves for a class of generalized Choquard equation with potentials, including harmonic cases. Suppose that $V(x)$ is a radial function and satisfies the the following conditions:

(V0). There exist two radial functions $V_1(x)$ and $V_2(x)$ such that $V(x) = V_1(x) + V_2(x)$.

(V1.1). $V_1(x) \in C^2(\mathbb{R}^3)$ and there exist positive constants m, M such that $0 \leq V_1(x)$ and $\leq M(1 + |x|^m)$ on \mathbb{R}^3 .

(V1.2). There exists $M_\alpha > 0$ such that $|x^\alpha \partial_x^\alpha V_1(x)| \leq M_\alpha(1 + V_1(x))$ on \mathbb{R}^3 for $|\alpha| \leq 2$.

(V1.3). $V_1(x) \in C^\infty(\mathbb{R}^3)$, $V_1(x)$ is positive in \mathbb{R}^3 and $\partial_x^\alpha V_1(x) \in L^\infty(\mathbb{R}^3)$ for $|\alpha| \geq 2$.

(V2). There exists $q \geq 1$ such that $q > 3/2$ and $x^\alpha \partial_x^\alpha V_2(x) \in L^q(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ for $|\alpha| \leq 2$.

Define the Hilbert space X by

$$X \triangleq \{v \in H^1(\mathbb{R}^3, \mathbb{C}); V_1(x)|v(x)|^2 \in L^1(\mathbb{R}^3)\}$$

with the inner product

$$(v, w)_X \triangleq \operatorname{Re} \int_{\mathbb{R}^3} (v(x)\overline{w(x)} + \nabla v(x) \cdot \overline{\nabla w(x)} + V_1(x)v(x)\overline{w(x)})dx$$

and the corresponding norm of X denoted by $\|\cdot\|_X$.

HSI1 **Proposition 1.1.** (Hardy-Littlewood-Sobolev inequality) Let $0 < \mu < n$ and suppose that $f \in L^q(\mathbb{R}^n), h \in L^r(\mathbb{R}^n)$ with $\frac{1}{q} + \frac{1}{r} + \frac{\mu}{n} = 2$ and $1 < q, r < \infty$, then

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x)||h(y)|}{|x-y|^\mu} dx dy \leq C(q, r, \mu, n) \|f\|_{L^q} \|h\|_{L^r}, \quad x, y \in \mathbb{R}^n,$$

where $C(q, r, \mu, n)$ is a positive constant depending on q, r, μ and n .

We define the energy functional E on X by

$$E(v) \triangleq \frac{1}{2} \|\nabla v\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)|v(x)|^2 dx - \frac{1}{2p} \mathfrak{F}_\mu(v)$$

with $\mathfrak{F}_\mu(v) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|v(x)|^p |v(y)|^p}{|x-y|^\mu} dx dy$. By assumptions (V0) – (V2) and $2 - \mu/3 < p < 6 - \mu$, applying Proposition 1.1, we know that $E(v)$ is well defined on X .

In the following we will make the following assumption.

CAUP **Proposition 1.2.** Let $0 < \mu < 3$. For any $u_0 \in X$, there exist $T = T(\|u_0\|_X) > 0$ and a unique solution $u(t) \in C([0, T], X)$ of (1.1) with $u(0) = u_0$ satisfying

$$E(u(t)) = E(u_0), \quad \|u(t)\|_2^2 = \|u_0\|_2^2, \quad t \in [0, T].$$

In addition, if $u_0 \in X$ satisfies $|x|u_0 \in L^2(\mathbb{R}^3)$, then the virial identity

$$\frac{d^2}{dt^2} \|xu(t)\|_2^2 = 8P(u(t)),$$

holds for $t \in [0, T]$, where

$$P(v) = \|\nabla v\|_2^2 - \frac{1}{2} \int_{\mathbb{R}^3} x \cdot \nabla V(x) |v(x)|^2 dx - \frac{3(p-2) + \mu}{2p} \mathfrak{F}_\mu(v). \quad (1.4) \quad \boxed{P}$$

In fact, for $V(x) = |x|^2$ and $0 < \mu < 3$, Chen et al. in [9] proved that

$$\frac{d}{dt} \|u(t)\|_2^2 = 0, \quad \frac{d}{dt} E(u(t)) = 0$$

and

$$\|u(t)\|_2^2 = \|u(0)\|_2^2 \quad \text{and} \quad E(u(t)) = E(u(0)).$$

Moreover, if $2 \leq p < 2 + (2 - \mu)/3$, the equation (1.1) exists globally in time for any initial value $u_0 \in X$; while for $2 \leq p = 2 + (2 - \mu)/3$, the equation (1.1) exists globally in time provided the initial data $\|u_0\|_{L^2}$ sufficiently small.

In order to state our results, we need to define on X ,

$$S_\omega(v) = E(v) + \omega Q(v), \quad Q(v) = \frac{1}{2}\|v\|_2^2$$

and

$$I_\omega(v) = \|\nabla v\|_2^2 + \omega\|v\|_2^2 + \int_{\mathbb{R}^3} V(x)|v(x)|^2 dx - \mathfrak{F}_\mu(v).$$

Consider the minimization problem as follows

$$S = \inf_{v \in \mathcal{N}} S_\omega(v), \tag{1.5}$$

where

$$\mathcal{N}_\omega = \{v \in X; v \neq 0, I_\omega(v) = 0\}. \tag{1.6} \quad \boxed{\text{A8}}$$

Definition 1.3. A ground state solution of (1.3) is $\varphi \in H^1(\mathbb{R}^3)$ with $v > 0$ and solving

$$S_\omega(\varphi) = \inf_{v \in \mathcal{N}_\omega} S_\omega(v). \tag{1.7} \quad \boxed{\text{B1}}$$

In the following we will use the notion

$$\mathcal{M}_\omega = \{v \in X; v \neq 0, S'_\omega(v) = 0, S_\omega(v) = S\} \tag{1.8} \quad \boxed{\text{A9}}$$

to denote the set of ground state solutions.

ppp **Remark 1.4.** We can assume that there exists $\omega_0 \in (0, \infty)$ such that \mathcal{M}_ω is not empty and $\mathcal{M}_\omega \subset \{v \in X_G; |x|v(x) \in L^2(\mathbb{R}^3) \text{ for any } \omega \in (\omega_0, \infty)\}$ (The detail we can see Section 2).

Now, we study the stability of the minimizers of (1.6) in the following sense.

Def **Definition 1.5.** For $\varphi_\omega \in \mathcal{M}_\omega$, we say that a standing wave solution $e^{i\omega t}\varphi_\omega(x)$ of (1.1) is stable in X if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\inf_{\theta \in \mathbb{R}} \|u(0) - e^{i\theta}\varphi_\omega\|_X < \delta$, $\theta \in \mathbb{R}$, then the solution $u(t)$ of (1.1) with $u(0) = u_0$ satisfies

$$\inf_{\theta \in \mathbb{R}} \|u(t) - e^{i\theta}\varphi_\omega\|_X < \varepsilon \quad \text{for any } t \geq 0.$$

Otherwise, $e^{i\omega t}\varphi_\omega$ is said to be unstable in X .

For instability of standing wave solution of (1.1), we have the following results.

Instability **Theorem 1.6.** Let $0 < \mu < 3$, $2 + (2 - \mu)/3 < p < 6 - \mu$ and assume that conditions (V0) – (V2) hold. Then there exists $\omega_0^* > \omega_0$ such that for any $\omega \in (\omega_0^*, \infty)$ with $\varphi_\omega(x) \in \mathcal{M}_\omega$, then the standing wave solution $u_\lambda(x, t) = e^{i\omega t}\varphi_\omega(x)$ of (1.1) is unstable in X .

The existence and qualitative properties of solutions of the Choquard equation have been widely studied in the last decades. In [33], Lieb proved the existence and uniqueness, up to translations, of the ground state of

$$-\Delta u + \omega u = (|x|^{-1} * |u|^2)u, \quad \text{in } \mathbb{R}^3.$$

Later, in [34], Lions showed the existence of a sequence of radially symmetric solutions. In [29, 21] the authors studied

$$-\Delta u + \omega u = (|x|^{-1} * |u|^p)|u|^{p-2}u, \quad \text{in } \mathbb{R}^3 \quad (1.9) \quad \boxed{\text{V0}}$$

showed the regularity, positivity and radial symmetry of the ground states and derived decay property at infinity as well. Generally, the uniqueness and nondegeneracy of the ground states is not known. In a recent paper [37], Xiang considered the uniqueness of the equation (1.9) and proved the following property of the ground state.

RGS **Lemma 1.7.** *There exists $0 < \eta < 1/3$ such that for any p , $2 < p < 2 + \eta$, there exists a unique positive radial ground state $\psi_1 \in H^1(\mathbb{R}^3)$ for equation*

$$-\Delta u + u = (|x|^{-1} * |u|^p)|u|^{p-2}u, \quad \text{in } \mathbb{R}^3. \quad (1.10) \quad \boxed{\text{UN}}$$

By Remark 1.4, we have the following stability results for the standing waves of equation (1.1).

MR **Theorem 1.8.** *Assume that conditions (V0) – (V2) hold and $2 < p < 2 + \eta'$ for some $0 < \eta' < \eta$ where $\eta > 0$ is the constant in Lemma 1.7. There exists $\omega_0^* > \omega_0$ such that, for any $\varphi_\omega(x) \in \mathcal{M}_\omega$, the standing wave solution $e^{i\omega t}\varphi_\omega(x)$ of (1.1) is stable in X_G in the sense of definition 1.5.*

In the last decades, many people studied the instability and stability of standing wave solution of local Schrödinger equation (see e.g. [2],[4],[10]–[15], [22],[23],[35],[36]):

$$i\partial_t u = -\Delta u + V(x)u + |u|^{p-1}u.$$

The idea of the present paper goes back to the paper [11, 13] by R. Fukuizumi, there the authors assumed that $V(x)$ satisfying conditions (V0) – (V2) and applied the concentration principle(see [19] and [20]) to study the ground state solution φ_ω of the following elliptic equation:

$$-\Delta \varphi + V(x)\varphi + \omega \varphi - |\varphi|^{p-1}\varphi = 0, \quad (1.11) \quad \boxed{\text{FEE}}$$

where $1 < p < 2^* - 1$. Then, $\tilde{\varphi}_\omega(x)$, defined by the scaling of $\varphi_\omega(x) = \omega^{1/(p-1)}\tilde{\varphi}_\omega(\sqrt{\omega}x)$, is a ground state solution of

$$-\Delta \varphi + \varphi + \omega^{-1}V\left(\frac{x}{\sqrt{\omega}}\right)\varphi - |\varphi|^{p-1}\varphi = 0. \quad (1.12) \quad \boxed{\text{REE2}}$$

Then under suitable assumptions on the $V(x)$, $\omega^{-1}V(\frac{x}{\sqrt{\omega}})\tilde{\varphi}_\omega \rightarrow 0$ in some sense as $\omega \rightarrow \infty$. Then for $p > 1 + 4/n$, inspired the behavior of the orbit of the standing wave

of (1.12) with $V(x) = 0$, he obtained that if the sufficient condition $\partial_\lambda^2 E(\varphi_\omega^\lambda)|_{\lambda=1} < 0$ holds, where $\varphi_\omega^\lambda = \lambda^{n/2} \varphi_\omega(\lambda x)$ then the ground state solution of (1.11) blows up at finite time, i.e. the standing wave solution is instable. Then, using the stability result of the standing wave of the limit problem, he proved that the standing wave of (1.11) is stable. This type of arguments was used by the authors in [5] to study the stability of the standing waves for a class of Hartree equation with potentials including the harmonic one as a particular case. In the present paper we follow the idea explored in [2] and [11] to study the generalized Choquard equation with a general class of potential which also includes the harmonic one as a particular case. Moreover, in our situation, the exponent p lies in a interval close to 2.

This paper is organized as follows. In section 2, we will prove some basic properties of the ground state solution of (1.1). In the first subsection of section 3, we give a sufficient condition for verifying the instability of standing wave solution of (1.1). Then, in subsection 3.2, we give the proof of main Theorem about of the instability. In subsection 4.1, the sufficient conditions for verifying the stability of standing wave solution of (1.1) with $\mu = 1$ is investigated. Finally, we prove the main result about of the stability.

2 Basic results for the ground states

In this section, we will give the definition of the ground state solution of (1.3) and prove the existence of the ground state solution of (1.3).

We study the functional $S_\omega(v) \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$ with its derivative given by

$$\langle S'_\omega(v), \phi \rangle = \int_{\mathbb{R}^3} (\nabla v \nabla \phi + \omega v \phi) dx + \int_{\mathbb{R}^3} V(x) v(x) \phi(x) dx - \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|v(x)|^{p-1} |v(y)|^p}{|x-y|^\mu} \phi(x) dx dy,$$

$\phi(x) \in C_0^\infty(\mathbb{R}^3)$. Hence, each critical point of $S_\omega(v)$ is a weak solution of (1.3). Let $W = \{v \in X; |x|v(x) \in L^2(\mathbb{R}^3)\}$ then it is easy to see that the embedding $W \hookrightarrow L^{q+1}$ is compact, where $1 \leq q < 5$. By variational arguments, we know

Proposition 2.1. *Let $0 < \mu < 3$ and $2 - \mu/3 < p < 2 + (2 - \mu)/3$. For $\omega > 0$, there exists $\varphi_\omega \in \mathcal{N}_\omega$ such that*

$$S_\omega(\varphi_\omega) = \inf_{v \in \mathcal{N}_\omega} S_\omega(v).$$

Then there is a Lagrange multiplier λ such that

$$S'_\omega(\varphi_\omega) - \lambda I'_\omega(\varphi_\omega) = 0. \tag{2.1} \quad \boxed{\text{B2}}$$

Multiple both side of the equation (2.1) by φ_ω , we obtain

$$\langle S'_\omega(\varphi_\omega), \varphi_\omega \rangle = \lambda \langle I'_\omega(\varphi_\omega), \varphi_\omega \rangle.$$

Noticing that

$$\langle S'_\omega(\varphi_\omega), \varphi_\omega \rangle = I_\omega(\varphi_\omega) = 0$$

and

$$\langle I'_\omega(\varphi_\omega), \varphi_\omega \rangle = -2(p-1)\mathfrak{F}_\mu(\varphi_\omega) < 0.$$

We know $\lambda = 0$ and φ_ω is a ground state solution of (1.3).

Inf **Lemma 2.2.** *Let $0 < \mu < 3$, $2 - \mu/3 < p < 6 - \mu$. For any $\omega > 0$ with $\varphi_\omega \in \mathcal{M}_\omega$, we have*

$$\begin{aligned}\mathfrak{F}_\mu(\varphi_\omega) &= \inf \left\{ \mathfrak{F}_\mu(v); v \in X \setminus \{0\}, I_\omega(v) = 0 \right\} \\ &= \inf \left\{ \mathfrak{F}_\mu(v); v \in X \setminus \{0\}, I_\omega(v) \leq 0 \right\},\end{aligned}$$

and

$$S_\omega(\varphi_\omega) = \inf \left\{ S_\omega(v); v \in X \setminus \{0\}, \mathfrak{F}_\mu(v) = \mathfrak{F}_\mu(\varphi_\omega) \right\}.$$

Proof. Since

$$S_\omega(v) = \frac{1}{2}I_\omega(v) + \frac{p-1}{2p}\mathfrak{F}_\mu(v), \quad v \in X,$$

we know that

$$\begin{aligned}\frac{p-1}{2p}\mathfrak{F}_\mu(\varphi_\omega) &= S_\omega(\varphi_\omega) = \inf \left\{ S_\omega(v); v \in X \setminus \{0\}, I_\omega(v) = 0 \right\} \\ &= \inf \left\{ \frac{p-1}{2p}\mathfrak{F}_\mu(v); v \in X \setminus \{0\}, I_\omega(v) = 0 \right\},\end{aligned}$$

i.e.

$$\mathfrak{F}_\mu(\varphi_\omega) = \inf \left\{ \mathfrak{F}_\mu(v); v \in X \setminus \{0\}, I_\omega(v) = 0 \right\}.$$

Let $\Gamma_\omega := \inf \left\{ \mathfrak{F}_\mu(v); v \in X \setminus \{0\}, I_\omega(v) \leq 0 \right\}$, it is obvious that

$$\Gamma_\omega \leq \mathfrak{F}_\mu(\varphi_\omega).$$

For any $v \in X \setminus \{0\}$ such that $I_\omega(v) < 0$, there exists $\lambda_0 \in (0, 1)$ satisfying $I_\omega(\lambda_0 v) = 0$. Consequently, we know that

$$\mathfrak{F}_\mu(\varphi_\omega) \leq \mathfrak{F}_\mu(\lambda_0 v) < \mathfrak{F}_\mu(v),$$

therefore

$$\mathfrak{F}_\mu(\varphi_\omega) = \Gamma_\omega.$$

For any v satisfying $\mathfrak{F}_\mu(v) = \mathfrak{F}_\mu(\varphi_\omega)$, it is easy to see that

$$I_\omega(v) \geq 0,$$

which implies that

$$S_\omega(v) \geq \frac{p-1}{2p}\mathfrak{F}_\mu(v) = S_\omega(\varphi_\omega) = S_\omega(\varphi_\omega).$$

Noticing that $\inf \left\{ S_\omega(v); v \in X \setminus \{0\}, \mathfrak{F}_\mu(v) = \mathfrak{F}_\mu(\varphi_\omega) \right\} \leq S_\omega(\varphi_\omega)$, we get the conclusion immediately. \square

ES **Lemma 2.3.** *Let $G(x) \in L^q(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ for some q such that $q > 3/2$ and $q \geq 1$. Then, there exists a constant $C > 0$ such that*

$$\left| \int_{\mathbb{R}^3} G(x) |v(x)|^2 dx \right| \leq C \|G\|_{L^q + L^\infty} \|v\|_{H^1}^2, \quad v \in H^1(\mathbb{R}^3).$$

Proof. The proof is very simple, we let $G(x) = g_1(x) + g_2(x)$, $g_1(x) \in L^q(\mathbb{R}^3)$, $g_2(x) \in L^\infty(\mathbb{R}^3)$ and use the Hölder inequality and Sobolev inequality to prove it. \square

3 Instability of standing waves

In the following, for any $\omega > 0$ with $\varphi_\omega \in \mathcal{M}_\omega$, we introduce the re-scaled function

$$\varphi_\omega(x) = \omega^{\frac{5-\mu}{4(p-1)}} \tilde{\varphi}_\omega(\sqrt{\omega}x). \quad (3.1) \quad \text{Res}$$

Then, $\tilde{\varphi}_\omega(x)$ is a ground state solution of

$$-\Delta \varphi + \varphi + \omega^{-1} V\left(\frac{x}{\sqrt{\omega}}\right) \varphi - (|x|^{-\mu} * |\varphi|^p) |\varphi|^{p-2} \varphi = 0. \quad (3.2) \quad \text{RRE}$$

Denote by

$$I_\omega^*(v) = \|\nabla v\|_2^2 + \|v\|_2^2 + \omega^{-1} \int_{\mathbb{R}^3} V\left(\frac{x}{\sqrt{\omega}}\right) |v(x)|^2 dx - \mathfrak{F}_\mu(v),$$

$$I_0(v) = \|\nabla v\|_2^2 + \|v\|_2^2 - \mathfrak{F}_\mu(v),$$

and let $\psi_1(x)$ be the ground state solution of

$$-\Delta \psi + \psi - (|x|^{-\mu} * |\psi|^p) |\psi|^{p-2} \psi = 0, \quad (3.3) \quad \text{Con1}$$

from [21], we know the regularity, positivity and radial symmetry of $\psi_1(x)$ and it decays asymptotically at infinity.

3.1 Sufficient conditions for instability

Pro **Lemma 3.1.** *Let $0 < \mu < 3$, $2 - \mu/3 < p < 6 - \mu$, $\varphi_\omega \in \mathcal{M}_\omega$ for large ω and assume that conditions (V0) – (V2) hold. Let $\tilde{\varphi}_\omega(x)$ be the re-scaled function defined by (3.1) and $\psi_1(x)$ be the ground state solution of (3.3). Then, we have*

- (1). $\lim_{\omega \rightarrow \infty} \mathfrak{F}_\mu(\tilde{\varphi}_\omega) = \mathfrak{F}_\mu(\psi_1)$;
- (2). $\lim_{\omega \rightarrow \infty} \omega^{-1} \int_{\mathbb{R}^3} V\left(\frac{x}{\sqrt{\omega}}\right) |\tilde{\varphi}_\omega(x)|^2 dx = 0$;
- (3). $\lim_{\omega \rightarrow \infty} \|\tilde{\varphi}_\omega\|_{H^1}^2 = \|\psi_1\|_{H^1}^2$.

Proof. (1). From Lemma 2.2, we know $\psi_1(x)$ is a minimizer of

$$\inf \left\{ \mathfrak{F}_\mu(v); v \in X \setminus \{0\}, I_0(v) \leq 0 \right\}. \quad (3.4) \quad \text{C2}$$

Similar to the arguments of Lemma 2.2, we may assume that $\tilde{\varphi}_\omega(x)$ is a minimizer of

$$\inf \left\{ \mathfrak{F}_\mu(v); v \in X \setminus \{0\}, I_\omega^*(v) \leq 0 \right\}. \quad (3.5) \quad \boxed{\text{C1}}$$

Notice that $I_0(\psi_1) = 0$, i.e.

$$\|\nabla \psi_1\|_2^2 + \|\psi_1\|_2^2 = \mathfrak{F}_\mu(\psi_1).$$

Then, for any $\theta > 1$, we have

$$\theta^{-2} I_\omega^*(\theta \psi_1) = -(\theta^{2p-2} - 1) \mathfrak{F}_\mu(\psi_1) + \omega^{-1} \int_{\mathbb{R}^3} V\left(\frac{x}{\sqrt{\omega}}\right) |\psi_1(x)|^2 dx. \quad (3.6) \quad \boxed{\text{C22}}$$

Using (V1.1) and Lemma 2.3, we know

$$\begin{aligned} & |\omega^{-1} \int_{\mathbb{R}^3} V\left(\frac{x}{\sqrt{\omega}}\right) |\psi_1(x)|^2 dx| \\ & \leq \omega^{-1} \int_{\mathbb{R}^3} V_1\left(\frac{x}{\sqrt{\omega}}\right) |\psi_1(x)|^2 dx + |\omega^{-1} \int_{\mathbb{R}^3} V_2\left(\frac{x}{\sqrt{\omega}}\right) |\psi_1(x)|^2 dx| \\ & \leq C\omega^{-1} \int_{\mathbb{R}^3} (1 + \omega^{-\frac{m}{2}} |x|^m) |\psi_1(x)|^2 dx + (\omega^{-1} + \omega^{\frac{3}{2q}-1}) C \|V_2\|_{L^q+L^\infty} \|\psi_1\|_{H^1}^2. \end{aligned}$$

Since $|x|^m |\psi_1(x)|^2 \in L^1(\mathbb{R}^3)$ and $q > 3/2$, using the fact that $\psi_1(x)$ decays exponentially at infinity, we get

$$\lim_{\omega \rightarrow \infty} \omega^{-1} \int_{\mathbb{R}^3} V\left(\frac{x}{\sqrt{\omega}}\right) |\psi_1(x)|^2 dx = 0. \quad (3.7) \quad \boxed{\text{C3}}$$

By (3.7) and (3.6), it is easy to see

$$\lim_{\omega \rightarrow \infty} \theta^{-2} I_\omega^*(\theta \psi_1) = \lim_{\omega \rightarrow \infty} -(\theta^{2p-2} - 1) \mathfrak{F}_\mu(\psi_1) < 0.$$

Namely, for any $\theta > 1$, if ω is large enough, we have

$$I_\omega^*(\theta \psi_1) < 0.$$

Next, since $I_\omega^*(\tilde{\varphi}_\omega) = 0$, i.e.

$$\|\nabla \tilde{\varphi}_\omega\|_2^2 + \|\tilde{\varphi}_\omega\|_2^2 = \mathfrak{F}_\mu(\tilde{\varphi}_\omega) - \omega^{-1} \int_{\mathbb{R}^3} V\left(\frac{x}{\sqrt{\omega}}\right) |\tilde{\varphi}_\omega(x)|^2 dx.$$

Then, for any $\theta > 1$, we have

$$\begin{aligned} \theta^{-2} I_0(\theta \tilde{\varphi}_\omega) &= -(\theta^{2p-2} - 1) \mathfrak{F}_\mu(\tilde{\varphi}_\omega) - \omega^{-1} \int_{\mathbb{R}^3} V\left(\frac{x}{\sqrt{\omega}}\right) |\tilde{\varphi}_\omega(x)|^2 dx \\ &\leq -(\theta^{2p-2} - 1) \mathfrak{F}_\mu(\tilde{\varphi}_\omega) + \omega^{-1} \int_{\mathbb{R}^3} V_-\left(\frac{x}{\sqrt{\omega}}\right) |\tilde{\varphi}_\omega(x)|^2 dx, \end{aligned} \quad (3.8) \quad \boxed{\text{C4}}$$

where $V_-(x) = \max\{-V(x), 0\}$. From the conditions (V0) – (V2), we have $V_- \in L^q + L^\infty$ with $q > 3/2$ and $q \geq 1$. According to Lemma 2.3, there exists $C > 0$ and $q > 3/2$ such that

$$\omega^{-1} \int_{\mathbb{R}^3} V\left(\frac{x}{\sqrt{\omega}}\right) |\tilde{\varphi}_\omega(x)|^2 dx \leq C(\omega^{\frac{3}{2q}-1} + \omega^{-1}) \|V_-\|_{L^q+L^\infty} \|\tilde{\varphi}_\omega\|_{H^1}^2. \quad (3.9) \quad \boxed{\text{C5}}$$

Using $I_\omega^*(\tilde{\varphi}_\omega) = 0$ again, we have

$$\begin{aligned}\|\tilde{\varphi}_\omega\|_{H^1}^2 &\leq \mathfrak{F}_\mu(\tilde{\varphi}_\omega) + \omega^{-1} \int_{\mathbb{R}^3} V_-(\frac{x}{\sqrt{\omega}}) |\tilde{\varphi}_\omega(x)|^2 dx \\ &\leq \mathfrak{F}_\mu(\tilde{\varphi}_\omega) + C(\omega^{\frac{3}{2q}-1} + \omega^{-1}) \|V_-\|_{L^q+L^\infty} \|\tilde{\varphi}_\omega\|_{H^1}^2,\end{aligned}$$

which implies

$$(1 - C(\omega^{\frac{3}{2q}-1} + \omega^{-1}) \|V_-\|_{L^q+L^\infty}) \|\tilde{\varphi}_\omega\|_{H^1}^2 \leq \mathfrak{F}_\mu(\tilde{\varphi}_\omega). \quad (3.10) \quad \boxed{\text{C6}}$$

According to (3.9) and (3.10), we have

$$\omega^{-1} \int_{\mathbb{R}^3} V_-(\frac{x}{\sqrt{\omega}}) |\tilde{\varphi}_\omega(x)|^2 dx \leq \frac{C(\omega^{\frac{3}{2q}-1} + \omega^{-1}) \|V_-\|_{L^q+L^\infty}}{1 - C(\omega^{\frac{3}{2q}-1} + \omega^{-1}) \|V_-\|_{L^q+L^\infty}} \mathfrak{F}_\mu(\tilde{\varphi}_\omega). \quad (3.11) \quad \boxed{\text{C7}}$$

Thus, from (3.8) and (3.11), we have

$$\theta^{-2} I_0(\theta \tilde{\varphi}_\omega) \leq -(\theta^{2p-2} - 1 - \frac{C(\omega^{\frac{3}{2q}-1} + \omega^{-1}) \|V_-\|_{L^q+L^\infty}}{1 - C(\omega^{\frac{3}{2q}-1} + \omega^{-1}) \|V_-\|_{L^q+L^\infty}}) \mathfrak{F}_\mu(\tilde{\varphi}_\omega).$$

Namely, for any $\theta > 1$, if ω is large enough, we have

$$I_0(\theta \tilde{\varphi}_\omega) < 0.$$

As stated above, we have

$$I_\omega^*(\theta \psi_1) < 0 \text{ and } I_0(\theta \tilde{\varphi}_\omega) < 0.$$

By $I_\omega^*(\theta \psi_1) < 0$ and (3.5), we have

$$\mathfrak{F}_\mu(\tilde{\varphi}_\omega) \leq \theta^{2p} \mathfrak{F}_\mu(\psi_1), \quad (3.12) \quad \boxed{\text{C8}}$$

while, by $I_0(\theta \tilde{\varphi}_\omega) < 0$ and (3.4), we have

$$\frac{1}{\theta^{2p}} \mathfrak{F}_\mu(\psi_1) \leq \mathfrak{F}_\mu(\tilde{\varphi}_\omega). \quad (3.13) \quad \boxed{\text{C9}}$$

Since $\theta > 1$ is arbitrary, from (3.12) and (3.13), we have

$$\lim_{\omega \rightarrow \infty} \mathfrak{F}_\mu(\tilde{\varphi}_\omega) = \mathfrak{F}_\mu(\psi_1).$$

(2). From $I_\omega^*(\tilde{\varphi}_\omega) = 0$, we have

$$-(\|\nabla \tilde{\varphi}_\omega\|_2^2 + \|\tilde{\varphi}_\omega\|_2^2 - \mathfrak{F}_\mu(\tilde{\varphi}_\omega)) = \omega^{-1} \int_{\mathbb{R}^3} V(\frac{x}{\sqrt{\omega}}) |\tilde{\varphi}_\omega(x)|^2 dx. \quad (3.14) \quad \boxed{\text{C99}}$$

Moreover, by $I_0(\theta \tilde{\varphi}_\omega) < 0$ with $\theta = 1$ and (1), we have

$$\limsup_{\omega \rightarrow \infty} I_0(\tilde{\varphi}_\omega) \leq 0. \quad (3.15) \quad \boxed{\text{C10}}$$

For $\omega \rightarrow \infty$, there exists $\theta(\omega) > 0$ such that $I_0(\theta(\omega)\tilde{\varphi}_\omega) = 0$, thus, we have

$$\mathfrak{F}_\mu(\psi_1) \leq \mathfrak{F}_\mu(\theta(\omega)\tilde{\varphi}_\omega) = \theta(\omega)^{2p} \mathfrak{F}_\mu(\tilde{\varphi}_\omega),$$

which together with conclusion (1) implies that

$$\liminf_{\omega \rightarrow \infty} \theta(\omega)^{2p} \geq \liminf_{\omega \rightarrow \infty} \frac{\mathfrak{F}_\mu(\psi_1)}{\mathfrak{F}_\mu(\tilde{\varphi}_\omega)} = 1.$$

Using $I_0(\theta(\omega)\tilde{\varphi}_\omega) = 0$ and conclusion (1) again, we can obtain

$$\liminf_{\omega \rightarrow \infty} I_0(\tilde{\varphi}_\omega) = \liminf_{\omega \rightarrow \infty} (\theta(\omega)^{2p-2} - 1) \mathfrak{F}_\mu(\tilde{\varphi}_\omega) \geq 0. \quad (3.16) \quad \boxed{\text{C11}}$$

From (3.15) and (3.16), we get

$$\lim_{\omega \rightarrow \infty} I_0(\tilde{\varphi}_\omega) = 0,$$

this implies that

$$\lim_{\omega \rightarrow \infty} \{\|\nabla \tilde{\varphi}_\omega\|_2^2 + \|\tilde{\varphi}_\omega\|_2^2 - \mathfrak{F}_\mu(\tilde{\varphi}_\omega)\} = 0. \quad (3.17) \quad \boxed{\text{C12}}$$

Hence, by (3.14), we get

$$\lim_{\omega \rightarrow \infty} \omega^{-1} \int_{\mathbb{R}^3} V\left(\frac{x}{\sqrt{\omega}}\right) |\tilde{\varphi}_\omega(x)|^2 dx = 0.$$

(3). From conclusion (1) and $I_0(\psi_1) = 0$, we have

$$\lim_{\omega \rightarrow \infty} \mathfrak{F}_\mu(\tilde{\varphi}_\omega) = \mathfrak{F}_\mu(\psi_1) = \|\psi_1\|_{H^1}^2.$$

By (3.17) in the proof of (2), we have

$$\lim_{\omega \rightarrow \infty} \|\tilde{\varphi}_\omega\|_{H^1}^2 = \lim_{\omega \rightarrow \infty} \{\|\nabla \tilde{\varphi}_\omega\|_2^2 + \|\tilde{\varphi}_\omega\|_2^2\} = \lim_{\omega \rightarrow \infty} \mathfrak{F}_\mu(\tilde{\varphi}_\omega).$$

Hence,

$$\lim_{\omega \rightarrow \infty} \|\tilde{\varphi}_\omega\|_{H^1}^2 = \|\psi_1\|_{H^1}^2.$$

□

PP **Proposition 3.2.** *Let $0 < \mu < 3$, $2 + (2 - \mu)/3 < p < 6 - \mu$ and assume that conditions (V0) – (V2) hold. Then there exists $\omega_0^* > \omega_0$ such that for any $\omega \in (\omega_0^*, \infty)$, $\varphi_\omega \in \mathcal{M}_\omega$,*

$$\partial_\lambda^2 E(\varphi_\omega^\lambda)|_{\lambda=1} < 0,$$

where, $\varphi_\omega^\lambda(x) = \lambda^{3/2} \varphi_\omega(\lambda x)$.

Proof. Let $\varphi_\omega^\lambda(x) = \lambda^{3/2} \varphi_\omega(\lambda x)$, by simple calculation, we have

$$E(\varphi_\omega^\lambda) = \frac{\lambda^2}{2} \|\nabla \varphi_\omega\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} V\left(\frac{x}{\lambda}\right) |\varphi_\omega(x)|^2 dx - \frac{\lambda^{3(p-2)+\mu}}{2p} \mathfrak{F}_\mu(\varphi_\omega)$$

and

$$\begin{aligned} \partial_\lambda^2 E(\varphi_\omega^\lambda)|_{\lambda=1} &= \|\nabla \varphi_\omega\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} \{2x \cdot \nabla V(x) + \sum_{i,j=1}^n x_i x_j \partial_i \partial_j V(x)\} |\varphi_\omega(x)|^2 dx \\ &\quad - \frac{\{3(p-2) + \mu\} \cdot \{3(p-2) + (\mu-1)\}}{2p} \mathfrak{F}_\mu(\varphi_\omega). \end{aligned}$$

Notice that φ_ω is ground state solution of (1.3), we have

$$P(\varphi_\omega) = \partial_\lambda S_\omega(\varphi_\omega^\lambda)|_{\lambda=1} = \langle S'_\omega(\varphi_\omega^\lambda), \partial_\lambda \varphi_\omega^\lambda|_{\lambda=1} \rangle = 0.$$

Thus, from the definition of (1.4), we have

$$\|\nabla \varphi_\omega\|_2^2 = \frac{1}{2} \int_{\mathbb{R}^3} x \cdot \nabla V(x) |v(x)|^2 dx + \frac{3(p-2) + \mu}{2p} \mathfrak{F}_\mu(\varphi_\omega).$$

Therefore,

$$\begin{aligned} \partial_\lambda^2 E(\varphi_\omega^\lambda)|_{\lambda=1} &= \frac{1}{2} \int_{\mathbb{R}^3} \{3x \cdot \nabla V(x) + \sum_{i,j=1}^3 x_i x_j \partial_i \partial_j V(x)\} |\varphi_\omega(x)|^2 dx \\ &\quad - \frac{\{3(p-2) + \mu\} \cdot \{3(p-2) + (\mu-2)\}}{2p} \mathfrak{F}_\mu(\varphi_\omega). \end{aligned} \quad (3.18) \quad \boxed{\text{C13}}$$

In the following, we set

$$V^*(x) = 3x \cdot \nabla V(x) + \sum_{i,j=1}^3 x_i x_j \partial_i \partial_j V(x)$$

and

$$V_k^* = 3x \cdot \nabla V_k(x) + \sum_{i,j=1}^3 x_i x_j \partial_i \partial_j V_k(x), \quad k = 1, 2,$$

with

$$V^*(x) = V_1^*(x) + V_2^*(x). \quad (3.19) \quad \boxed{\text{C14}}$$

By Lemma 2.3, (2), (3) of Lemma 3.1 and condition (V2), we have

$$\omega^{-1} \int_{\mathbb{R}^3} |V_2(\frac{x}{\sqrt{\omega}})| \tilde{\varphi}_\omega(x)|^2 dx \leq C(\omega^{\frac{3}{2q}-1} + \omega^{-1}) \|V_2\|_{L^q+L^\infty} \|\tilde{\varphi}_\omega\|_{H^1}^2 \quad (3.20) \quad \boxed{\text{C15}}$$

and

$$\lim_{\omega \rightarrow \infty} \omega^{-1} \int_{\mathbb{R}^3} V_1(\frac{x}{\sqrt{\omega}}) |\tilde{\varphi}_\omega(x)|^2 dx = 0. \quad (3.21) \quad \boxed{\text{C16}}$$

Moreover, from the condition (V1.2), we have

$$\omega^{-1} \int_{\mathbb{R}^3} |V_1^*(\frac{x}{\sqrt{\omega}})| |\tilde{\varphi}_\omega(x)|^2 dx \leq C\omega^{-1} \int_{\mathbb{R}^3} (1 + V_1(\frac{x}{\sqrt{\omega}})) |\tilde{\varphi}_\omega(x)|^2 dx.$$

Thus, from (3.21) and Lemma 3.2 (3), we have

$$\lim_{\omega \rightarrow \infty} \omega^{-1} \int_{\mathbb{R}^3} |V_1^*(\frac{x}{\sqrt{\omega}})| |\tilde{\varphi}_\omega(x)|^2 dx = 0. \quad (3.22) \quad \boxed{\text{C17}}$$

And the same as (3.20), we still have

$$\omega^{-1} \int_{\mathbb{R}^3} |V_2^*(\frac{x}{\sqrt{\omega}})| \tilde{\varphi}_\omega(x)^2 dx \leq C(\omega^{\frac{3}{2q}-1} + \omega^{-1}) \|V_2^*\|_{L^q+L^\infty} \|\tilde{\varphi}_\omega\|_{H^1}^2.$$

and

$$\lim_{\omega \rightarrow \infty} \omega^{-1} \int_{\mathbb{R}^3} |V_2^*(\frac{x}{\sqrt{\omega}})| \tilde{\varphi}_\omega(x)^2 dx = 0. \quad (3.23) \quad \boxed{\text{C18}}$$

According to (3.19), (3.22) and (3.23), we have

$$\lim_{\omega \rightarrow \infty} \omega^{-1} \int_{\mathbb{R}^3} |V^*(\frac{x}{\sqrt{\omega}})| \tilde{\varphi}_\omega(x)^2 dx = 0. \quad (3.24) \quad \boxed{\text{C19}}$$

From (1) of Lemma 3.1, (3.24) and the definition of $\tilde{\varphi}_\omega(x)$, we have

$$\lim_{\omega \rightarrow \infty} \frac{\omega^{-1} \int_{\mathbb{R}^3} V^*(\frac{x}{\sqrt{\omega}}) |\tilde{\varphi}_\omega|^2 dx}{\mathfrak{F}_\mu(\tilde{\varphi}_\omega)} = \lim_{\omega \rightarrow \infty} \frac{\int_{\mathbb{R}^3} V^*(x) |\varphi_\omega|^2 dx}{\mathfrak{F}_\mu(\varphi_\omega)} = 0. \quad (3.25) \quad \boxed{\text{C20}}$$

Since $\varphi_\omega(x) = \omega^{\frac{5-\mu}{4(p-1)}} \tilde{\varphi}_\omega(\sqrt{\omega}x)$, we have $\tilde{\varphi}_\omega(x) = \omega^{\frac{\mu-5}{4(p-1)}} \varphi_\omega(\frac{x}{\sqrt{\omega}})$.

$$\frac{\omega^{-1} \int_{\mathbb{R}^3} V^*(\frac{x}{\sqrt{\omega}}) |\tilde{\varphi}_\omega|^2 dx}{\mathfrak{F}_\mu(\tilde{\varphi}_\omega)} = \frac{\omega^{\frac{\mu-5}{2(p-1)}} \omega^{-1} \omega^{\frac{3}{2}} \int_{\mathbb{R}^3} V^*(x) |\varphi_\omega|^2 dx}{\omega^{\frac{(\mu-5)p}{2(p-1)}} \omega^3 \omega^{-\frac{\mu}{2}} \mathfrak{F}_\mu(\varphi_\omega)} = \frac{\int_{\mathbb{R}^3} V^*(x) |\varphi_\omega|^2 dx}{\mathfrak{F}_\mu(\varphi_\omega)}$$

Since $p > 2 + (2 - \mu)/3$, we have

$$\frac{\{3(p-2) + \mu\} \cdot \{3(p-2) + (\mu-2)\}}{2p} > 0. \quad (3.26) \quad \boxed{\text{C21}}$$

By (3.25) and (3.26), we have

$$\frac{\int_{\mathbb{R}^3} V^*(x) |\varphi_\omega|^2 dx}{\mathfrak{F}_\mu(\varphi_\omega)} < \frac{\{3(p-2) + \mu\} \cdot \{3(p-2) + (\mu-2)\}}{2p}, \quad (3.27) \quad \boxed{\text{C221}}$$

if ω is large enough. From (3.18) and (3.27) we know there exists $\omega_0^* > \omega_0$ such that for any $\omega \in (\omega_0^*, \infty)$,

$$\partial_\lambda^2 E(\varphi_\omega^\lambda)|_{\lambda=1} < 0.$$

□

3.2 Proof of the main result

In this section, we are going to give the proof of the main result.

For any $\varphi_\omega \in X$ and $\varepsilon > 0$, we define

$$U_\varepsilon(\varphi_\omega) \triangleq \{v \in X; \inf_{\theta \in \mathbb{R}} \|v - e^{i\theta} \varphi_\omega\|_X < \varepsilon\}.$$

31 **Lemma 3.3.** *Let φ_ω be a ground state solution of (1.3). If $\partial_\lambda^2 E(\varphi_\omega^\lambda)|_{\lambda=1} < 0$, then there exists $\varepsilon > 0$, $\delta > 0$ and mapping $\lambda : U_\varepsilon(\varphi_\omega) \rightarrow (1 - \delta, 1 + \delta)$ such that*

$$I(v^{\lambda(v)}) = 0 \quad \text{for all } v \in U_\varepsilon(\varphi_\omega).$$

Proof. Let

$$F(v, \lambda) = I(v^\lambda).$$

Since φ_ω is a minimizer of $S_\omega(v)$ constrained on the manifold \mathcal{N}_ω , then

$$\langle S_\omega''(\varphi_\omega)\phi, \phi \rangle \geq 0, \quad \text{for } \langle \varphi_\omega, \phi \rangle = 0. \quad (3.28) \quad \boxed{\text{D1}}$$

Next, take $\eta = \partial_\lambda \varphi_\omega^\lambda|_{\lambda=1}$, since

$$Q(\varphi_\omega) = Q(\varphi_\omega^\lambda) \quad \text{and} \quad \langle S_\omega'(\varphi_\omega), \xi \rangle = 0, \forall \xi \in X,$$

then

$$\langle S_\omega''(\varphi_\omega)\eta, \eta \rangle = \partial_\lambda^2 E(\varphi_\omega^\lambda)|_{\lambda=1} < 0. \quad (3.29) \quad \boxed{\text{D2}}$$

By (3.28) and (3.29), we know $\langle \eta, \varphi_\omega \rangle \neq 0$ and so

$$\partial_\lambda F(\varphi_\omega, 1) = \partial_\lambda I(\varphi_\omega^\lambda)|_{\lambda=1} = \langle I'(\varphi_\omega), \eta \rangle \neq 0.$$

Notice that,

$$F_\lambda(\varphi_\omega, 1) = I(\varphi_\omega) = 0,$$

the implicit function theorem implies the existence of $\varepsilon > 0$, $\delta > 0$ and a mapping $\lambda : B_\varepsilon(\varphi_\omega) \rightarrow (1 - \delta, 1 + \delta)$ such that

$$I(v^{\lambda(v)}) = 0 \quad \text{for all } v \in B_\varepsilon(\varphi_\omega),$$

the conclusion then follows directly. \square

32 **Lemma 3.4.** *If $\partial_\lambda^2 E(\varphi_\omega^\lambda)|_{\lambda=1} < 0$, where φ_ω is a ground state solution of (1.3), then there exists $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that, for any $v \in U_{\varepsilon_0}(\varphi_\omega)$ satisfying $\|v\|_2^2 = \|\varphi_\omega\|_2^2$, we have*

$$E(\varphi_\omega) \leq E(v) + (\lambda(v) - 1)P(v),$$

for some $\lambda(v) \in (1 - \delta_0, 1 + \delta_0)$.

Proof. Since $\partial_\lambda^2 E(\varphi_\omega^\lambda)|_{\lambda=1} < 0$ and $\partial_\lambda^2 E(v^\lambda)$ is continuous in λ and v , we know that there exists $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that $\partial_\lambda^2 E(v^\lambda) < 0$ for any $\lambda \in (1 - \delta_0, 1 + \delta_0)$ and $v \in U_{\varepsilon_0}(\varphi_\omega)$. Notice that $\partial_\lambda E(v^\lambda)|_{\lambda=1} = P(v)$, applying the Taylor expansion for the function $E(v^\lambda)$ at $\lambda = 1$, we have

$$E(v^\lambda) \leq E(v) + (\lambda - 1)P(v), \quad \lambda \in (1 - \delta_0, 1 + \delta_0), \quad v \in U_{\varepsilon_0}(\varphi_\omega). \quad (3.30) \quad \boxed{\text{D3}}$$

By Lemma 3.3, we choose $\varepsilon_0 < \varepsilon$ and $\delta_0 < \delta$, then there exists $\lambda(v) \in (1 - \delta_0, 1 + \delta_0)$ such that

$$I(v^{\lambda(v)}) = 0, \quad \forall v \in U_{\varepsilon_0}(\varphi_\omega).$$

Therefore, we have

$$S_\omega(v^{\lambda(v)}) \geq S_\omega(\varphi_\omega).$$

Since $\|v^{\lambda(v)}\|_2^2 = \|v\|_2^2 = \|\varphi_\omega\|_2^2$, we obtain

$$E(v^{\lambda(v)}) = S_\omega(v^{\lambda(v)}) - \frac{\omega}{2}\|v^{\lambda(v)}\|_2^2 \geq S_\omega(\varphi_\omega) - \frac{\omega}{2}\|\varphi_\omega\|_2^2 = E(\varphi_\omega). \quad (3.31) \quad \boxed{\text{D4}}$$

Thus, from (3.30) and (3.31), we obtain

$$E(\varphi_\omega) \leq E(v) + (\lambda(v) - 1)P(v), \quad \forall v \in U_{\varepsilon_0}(\varphi_\omega).$$

□

Let φ_ω be a ground state solution of (1.3) in Lemma 3.4, we introduce

$$\mathcal{K}_\omega \triangleq \{v \in U_{\varepsilon_0}(\varphi_\omega); E(v) < E(\varphi_\omega), \|v\|_2^2 = \|\varphi_\omega\|_2^2, P(v) < 0\},$$

and

$$T(u_0) = \sup\{T > 0; u(t) \in U_{\varepsilon_0}(\varphi_\omega), 0 \leq t \leq T\},$$

where $u(t)$ is a solution of (1.1) with $u(0) = u_0$. Then, we have the following Lemma.

33 Lemma 3.5. *If $\partial_\lambda^2 E(\varphi_\omega^\lambda)|_{\lambda=1} < 0$, then for any $u_0 \in \mathcal{K}_\omega$, there exists $\delta_2 = \delta_2(u_0) > 0$ such that $P(u(t)) \leq -\delta_2$ for $0 \leq t \leq T(u_0)$.*

Proof. Take $u_0 \in \mathcal{K}_\omega$ and put $\delta_1 = E(\varphi_\omega) - E(u_0) > 0$. From Lemma 3.4 and $E(u(t)) = E(u_0)$, we have

$$E(\varphi_\omega) \leq E(u(t)) + (\lambda(u(t)) - 1)P(u(t)) = E(u_0) + (\lambda(u(t)) - 1)P(u(t)),$$

which implies

$$0 < \delta_1 \leq (\lambda(u(t)) - 1)P(u(t)), \quad 0 \leq t < T(u_0). \quad (3.32) \quad \boxed{\text{D5}}$$

Thus, $P(u(t)) \neq 0$. Since $u_0 \in \mathcal{K}_\omega$ then $P(u_0) < 0$. By the continuous of $P(u(t))$ in t , we know

$$P(u(t)) < 0 \quad \text{for } 0 \leq t < T(u_0). \quad (3.33) \quad \boxed{\text{D6}}$$

Then $\lambda(u(t)) \in (1 - \delta_0, 1)$, from (3.32) and (3.33), we have

$$P(u(t)) \leq \frac{\delta_1}{\lambda(u(t)) - 1} \leq -\frac{\delta_1}{\delta_0}, \quad 0 \leq t < T(u_0).$$

Hence, let $\delta_2 = \delta_1/\delta_0$, we have $P(u(t)) \leq -\delta_2$ for $0 \leq t < T(u_0)$. □

Proof of Theorem 1.6. Since $P(\varphi_\omega) = \partial_\lambda S_\omega(\varphi_\omega^\lambda)|_{\lambda=1} = \partial_\lambda E(\varphi_\omega^\lambda)|_{\lambda=1} = 0$ and $\partial_\lambda^2 E(\varphi_\omega^\lambda)|_{\lambda=1} < 0$, there exists $\delta > 0$ such that

$$E(\varphi_\omega^\lambda) < E(\varphi_\omega) \quad \text{for } \lambda \in (1, 1 + \delta_0).$$

On the other hand, since

$$P(\varphi_\omega^\lambda) = \lambda \partial_\lambda E(\varphi_\omega^\lambda),$$

we have

$$\partial_\lambda P(\varphi_\omega^\lambda)|_{\lambda=1} = P(\varphi_\omega) + \partial_\lambda^2 E(\varphi_\omega^\lambda)|_{\lambda=1} = \partial_\lambda^2 E(\varphi_\omega^\lambda)|_{\lambda=1} < 0.$$

Moreover, we have

$$\|\varphi_\omega^\lambda\|_2^2 = \|\varphi_\omega\|_2^2 \quad \text{and} \quad \lim_{\lambda \rightarrow 1} \|\varphi_\omega^\lambda - \varphi_\omega\|_X = 0.$$

Therefore, we have $\varphi_\omega^\lambda \in \mathcal{K}_\omega$ for $\lambda > 1$ sufficiently close to 1.

Since we have $|x|\varphi_\omega^\lambda(x) \in L^2(\mathbb{R}^3)$, from Assumption ??, we have

$$\frac{d^2}{dt^2} \|xu_\lambda(t)\|_2^2 = 8P(u_\lambda(t)), \quad 0 \leq t < T(\varphi_\omega^\lambda),$$

where $u_\lambda(t)$ is the solution of (1.1) with $u_\lambda(0) = \varphi_\omega^\lambda$.

By Lemma 3.5, there exists $\delta_\lambda > 0$ such that

$$P(u_\lambda(t)) \leq -\delta_\lambda, \quad 0 \leq t < T(\varphi_\omega^\lambda).$$

Set $g(t) = \|xu_\lambda(t)\|_2^2 > 0$, the Taylor expansion at $t = 0$ gives

$$g(t) \leq g(0) + g'(0)t + \frac{t^2}{2}g''(0) \leq g(0) + g'(0)t - \frac{\delta_\lambda}{2}t^2 \quad \text{for } 0 \leq t < T(\varphi_\omega^\lambda). \quad (3.34) \quad \boxed{\text{D8}}$$

This implies $T(\varphi_\omega^\lambda) < \infty$. Otherwise, if $T(\varphi_\omega^\lambda) = \infty$, by (3.34), there exists T_1 such that $g(T_1) < 0$, this contradicts $g(t) \geq 0$ for any $t \in [0, T(\varphi_\omega^\lambda))$.

Combining this with the fact that $|x|\varphi_\omega^\lambda = |x|u_\lambda(0) \in L^2(\mathbb{R}^3)$, there exists a $T_0 > 0$ such that

$$\lim_{t \rightarrow T_0^-} \|xu_\lambda(t)\|_2^2 = 0.$$

Moreover, we have

$$\int_{\mathbb{R}^3} |u_\lambda|^2 \leq C \left(\int_{\mathbb{R}^3} |x|^2 |u_\lambda|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\nabla u_\lambda|^2 \right)^{\frac{1}{2}},$$

and C is independent of u_λ , by the conservation of mass $\|u_\lambda\|_2^2 = \|\varphi_\omega^\lambda\|_2^2 = \text{const} > 0$, we have

$$\lim_{t \rightarrow T_0^-} \|\nabla u_\lambda\|_2^2 = +\infty,$$

and

$$\|u_\lambda\|_{H^1}^2 = \|u_\lambda\|_2^2 + \|\nabla u_\lambda\|_2^2 = +\infty.$$

4 Stability of standing waves

For $2 - \mu/3 < p < 2 + (2 - \mu)/3$, the uniqueness of the ground state of (1.9) is not known. We consider the stability of standing wave for (1.1) with $\mu = 1$ when p is sufficiently closed to 2. When $\mu = 1$ in (3.3), the uniqueness of $\psi_1(x)$ was investigated in [37]. Denoted by

$$\begin{aligned} I_0(\varphi) &= \|\nabla \varphi\|_2^2 + \|\varphi\|_2^2 - \mathfrak{F}_1(\varphi), & F_0(\varphi) &= \frac{1}{2p} \mathfrak{F}_1(\varphi), \\ S_0(\varphi) &= \frac{1}{2} \|\nabla \varphi\|_2^2 + \frac{1}{2} \|v\|_2^2 - \frac{1}{2p} \mathfrak{F}_1(\varphi). \end{aligned} \quad (4.1) \quad \boxed{\text{F}}$$

4.1 Sufficient conditions for the orbital stability

In the following, we show ψ_1 is the positive minimizer of

$$\begin{aligned} S_1 &= \inf\{S_0(\varphi); \varphi \in H^1 \setminus \{0\}, I_0(\varphi) = 0\} \\ &= \inf\{F_0(\varphi); \varphi \in H^1 \setminus \{0\}, I_0(\varphi) = 0\}. \end{aligned}$$

Next, we need a important result as follows.

CP **Lemma 4.1.** *Assume that p satisfies the assumption of Lemma 1.7 and let ψ_1 is the unique positive and radially symmetric solution of*

$$-\Delta\psi + \psi - (|x|^{-1} * |\psi|^p)|\psi|^{p-2}\psi = 0,$$

then ψ_1 is the minimizer of the following variational problem

$$S_1 = \inf\{F_0(\varphi); \varphi \in H^1 \setminus \{0\}, I_0(\varphi) = 0\}. \quad (4.2) \quad \text{JH}$$

Then, there exists a subsequence $\{\varphi_k\}$, if $I_0(\varphi_k) \rightarrow 0$ and $F_0(\varphi_k) \rightarrow S_1$ as $k \rightarrow \infty$, then there exists a sequence $\{y_k\} \subset \mathbb{R}^3$ satisfying

$$\lim_{k \rightarrow \infty} \|\varphi_k(\cdot + y_k) - \psi_1\|_{H^1} = 0.$$

Proof. Let $\{\varphi_k\} \subset H^1 \setminus \{0\}$ be a bounded sequence such that $I_0(\varphi_k) \rightarrow 0$ and $F_0(\varphi_k) \rightarrow S_1$ as $k \rightarrow \infty$. We have the decomposition property for a subsequence $\{\varphi_k\} \subset H^1 \setminus \{0\}$: there exists a sequence $\{\varphi^j\}$ in H^1 , for any $l \geq 1$, we have the following identity

$$\varphi_k(x) = \sum_{j=1}^l \varphi^j(x - x_k^j) + \varphi_k^l(x), \quad (4.3)$$

with $\lim_{k \rightarrow \infty} \|\varphi_k^l\|_p \rightarrow 0$ as $l \rightarrow \infty$ and for every $j_1 \neq j_2$, $|x_k^{j_1} - x_k^{j_2}| \rightarrow \infty$ as $k \rightarrow \infty$. Moreover, as $k \rightarrow \infty$, we have

$$\|\varphi_k\|_{H^1}^2 = \sum_{j=1}^l \|\varphi^j\|_{H^1}^2 + \|\varphi_k^l\|_{H^1}^2 + o(1). \quad (4.4) \quad \text{FJ}$$

By (4.1), we know

$$\|\varphi_k\|_{H^1}^2 = I_0(\varphi_k) + 2pF_0(\varphi_k). \quad (4.5) \quad \text{PM}$$

From (4.4),(4.5) and $k \rightarrow \infty$, we have

$$I_0(\varphi_k) = \sum_{j=1}^l I_0(\varphi^j) + 2p \sum_{j=1}^l F_0(\varphi^j) + \|\varphi_k^l\|_{H^1}^2 - 2pF_0(\varphi_k) + o(1). \quad (4.6) \quad \text{DHS}$$

Thus we have

$$\sum_{j=1}^l I_0(\varphi^j) + 2p \sum_{j=1}^l F_0(\varphi^j) - 2pS_1 \leq 0, \quad (4.7) \quad \text{IEQ}$$

and

$$\lim_{k \rightarrow \infty} F_0(\varphi_k) = \sum_{j=1}^{\infty} F_0(\varphi^j). \quad (4.8) \quad \boxed{\text{JXS}}$$

According to (4.7) and (4.8), we derive

$$\sum_{j=1}^{\infty} I_0(\varphi^j) \leq 0. \quad (4.9) \quad \boxed{\text{XYL}}$$

We claim that there exists exactly one j such that φ^j is nonzero. Suppose this is true, we may assume that $j = 1$, then, we have

$$\varphi_k(x) = \varphi^1(x - x_k^1) + \varphi_k^1, \quad (4.10) \quad \boxed{\text{ZYF}}$$

with $\lim_{k \rightarrow \infty} \|\varphi_k^1\|_p \rightarrow 0$ and $I_0(\varphi^1) \leq 0$. We may derive $I_0(\varphi^1) = 0$. In fact, if $\varphi^1 \neq 0$ and $I_0(\varphi^1) < 0$, then there exists some $\lambda_0 \in (0, 1)$ such that $I_0(\lambda_0 \varphi^1) = 0$. Then, by (4.2), we have $2pS_1 \leq 2pF_0(\lambda_0 \varphi^1) = \|\lambda_0 \varphi^1\|_{H^1}^2 < \|\varphi^1\|_{H^1}^2$. However, from (4.4), we know $\|\varphi^1\|_{H^1}^2 \leq 2pS_1$, this is a contradiction. From (4.4), we know $\lim_{k \rightarrow \infty} \|\varphi_k^1\|_{H^1} = 0$. Therefore,

$$\varphi_k(\cdot + x_k^1) \rightarrow \varphi^1, \quad \text{in } H^1(\mathbb{R}^3),$$

with φ^1 being a minimizer of (4.2). On the other hand, we know $|\varphi^1|$ is also a minimizer of (4.2) by Kato's inequality $|\nabla|\varphi^1|| \leq |\nabla\varphi^1|$. From [21], we know $|\varphi^1|$ is radially symmetric up to shifting the origin and a $H^1(\mathbb{R}^3)$ solution to

$$-\Delta\psi + \psi - (|x|^{-1} * |\psi|^p)|\psi|^{p-2}\psi = 0 \quad (4.11) \quad \boxed{\text{Tem}}$$

with the least energy. Since ψ_1 is the unique positive, radial solution of (4.11), we know there exists y such that

$$|\varphi^1|(x - y) = \psi_1.$$

Let $y_k = x_k^1 - y$, then we know

$$\lim_{k \rightarrow \infty} \|\varphi_k(\cdot + y_k) - \psi_1\|_{H^1} = 0.$$

Now we are ready to prove the claim that there exists exactly one j such that φ^j is nonzero. First repeat the same arguments as above, we can show that for every $j \geq 0$, $I_0(\varphi^j) = 0$. Next if there exists two $\varphi^j \neq 0$ (denoted by φ^{j_1} and φ^{j_2}). By (4.4), we know $\|\varphi^{j_i}\|_{H^1}^2 < 2pS_1$. But, (4.2) implies that $2pS_1 \leq 2pF_0(\varphi^{j_i}) = \|\varphi^{j_i}\|_{H^1}^2$, this is still a contradiction. Thus the claim is proved. \square

Lemma 4.2. *Suppose that (V0) – (V2) are satisfied and p satisfies the assumption of Lemma 1.7. Let $\varphi_\omega(x) \in \mathcal{M}_\omega$ and the unique positive radial solution ψ_1 of (1.9) with $\omega = 1$. Then, for the re-scaled function $\tilde{\varphi}_\omega(x)$ defined by*

$$\varphi_\omega(x) = \omega^{\frac{1}{p-1}} \tilde{\varphi}_\omega(\sqrt{\omega}x),$$

we have

$$\lim_{\omega \rightarrow \infty} \|\tilde{\varphi}_\omega - \psi_1\|_{H^1} = 0.$$

Proof. We introduce two functional as

$$\begin{aligned} I_\omega^*(\varphi) &= \|\nabla \varphi\|_2^2 + \|\varphi\|_2^2 + \omega^{-1} \int_{\mathbb{R}^3} V\left(\frac{x}{\sqrt{\omega}}\right) |\varphi|^2 - \mathfrak{F}_1(\varphi), \\ S_\omega^*(\varphi) &= \frac{1}{2} \|\nabla \varphi\|_2^2 + \frac{1}{2} \|\varphi\|_2^2 + \frac{1}{2} \omega^{-1} \int_{\mathbb{R}^3} V\left(\frac{x}{\sqrt{\omega}}\right) |\varphi|^2 - \frac{1}{2p} \mathfrak{F}_1(\varphi). \end{aligned}$$

For fixed $\omega > 0$, we claim that $\tilde{\varphi}_\omega$ is a minimizer of the variational problem as follows,

$$S^* = \inf \left\{ \frac{p-1}{2p} \mathfrak{F}_1(\varphi); \varphi \in H^1 \setminus \{0\}, I_\omega^*(\varphi) \leq 0 \right\}. \quad (4.12) \quad \boxed{\text{S1}}$$

In fact, since

$$S_\omega(v) = \frac{1}{2} I_\omega(v) + \frac{p-1}{2p} \mathfrak{F}_1(v), \quad v \in X,$$

we know that

$$\begin{aligned} \frac{p-1}{2p} \mathfrak{F}_1(\varphi_\omega) &= S_\omega(\varphi_\omega) = \inf \left\{ S_\omega(v); v \in X \setminus \{0\}, I_\omega(v) = 0 \right\} \\ &= \inf \left\{ \frac{p-1}{2p} \mathfrak{F}_1(v); v \in X \setminus \{0\}, I_\omega(v) = 0 \right\}, \end{aligned}$$

i.e.

$$\mathfrak{F}_1(\varphi_\omega) = \inf \left\{ \mathfrak{F}_1(v); v \in X \setminus \{0\}, I_\omega(v) = 0 \right\}.$$

Let $\Gamma_\omega := \inf \left\{ \mathfrak{F}_1(v); v \in X \setminus \{0\}, I_\omega(v) \leq 0 \right\}$, it is obvious that

$$\Gamma_\omega \leq \mathfrak{F}_1(\varphi_\omega).$$

For any $v \in X \setminus \{0\}$ such that $I_\omega(v) < 0$, there exists $\lambda_0 \in (0, 1)$ satisfying $I_\omega(\lambda_0 v) = 0$. Consequently, we know that

$$\mathfrak{F}_1(\varphi_\omega) \leq \mathfrak{F}_1(\lambda_0 v) < \mathfrak{F}_\mu(v),$$

therefore

$$\mathfrak{F}_1(\varphi_\omega) = \Gamma_\omega.$$

Consequently, by changing variable, we know $\tilde{\varphi}_\omega$ minimizes (4.12). Similarly, ψ_1 is the minimizer of

$$\inf \left\{ \frac{p-1}{2p} \mathfrak{F}_\mu(\varphi); \varphi \in H^1 \setminus \{0\}, I_0(\varphi) \leq 0 \right\}. \quad (4.13) \quad \boxed{\text{S2}}$$

From Lemma 3.1 (1), we have

$$\lim_{\omega \rightarrow \infty} \mathfrak{F}_1(\tilde{\varphi}_\omega) = \mathfrak{F}_1(\psi_1). \quad (4.14) \quad \boxed{\text{equality1}}$$

Noting that $I_\omega^*(\tilde{\varphi}_\omega) = 0$, we know

$$I_0(\tilde{\varphi}_\omega) = \|\nabla \tilde{\varphi}_\omega\|_2^2 + \|\tilde{\varphi}_\omega\|_2^2 - \mathfrak{F}_1(\tilde{\varphi}_\omega) = -\omega^{-1} \int_{\mathbb{R}^3} V\left(\frac{x}{\sqrt{\omega}}\right) |\tilde{\varphi}_\omega|^2 < 0. \quad (4.15) \quad \boxed{\text{inequality1}}$$

Hence, there exists a $\lambda_0(\omega) \in (0, 1]$ such that $I_0(\lambda_0(\omega)\tilde{\varphi}_\omega) = 0$, then, we have

$$\mathfrak{F}_1(\psi_1) \leq \lambda_0(\omega)^{2p} \mathfrak{F}_1(\tilde{\varphi}_\omega) \rightarrow \lambda_0^{2p} \mathfrak{F}_1(\psi_1), \quad \text{as } \omega \rightarrow \infty,$$

for some $\lambda_0 \in (0, 1]$, which implies $\lim_{\omega \rightarrow \infty} \lambda_0(\omega) = 1$. Thus, $\lim_{\omega \rightarrow \infty} I_0(\tilde{\varphi}_\omega) = 0$, we may get from (4.15) that

$$\lim_{\omega \rightarrow \infty} \omega^{-1} \int_{\mathbb{R}^3} V\left(\frac{x}{\sqrt{\omega}}\right) |\tilde{\varphi}_\omega|^2 = 0.$$

By Lemma 4.1 for any sequence $\{\omega_k\}$ with $\omega_k \rightarrow \infty$, there exists a subsequence of $\{\tilde{\varphi}_{\omega_k}\}$ and a sequence $\{y_k\} \subset \mathbb{R}^3$ such that

$$\lim_{k \rightarrow \infty} \|\tilde{\varphi}_{\omega_k}(\cdot + y_k) - \psi_1\|_{H^1} = 0. \quad (4.16) \quad \boxed{\text{CCP}}$$

Since $\tilde{\varphi}_{\omega_k} \in X_G$ and the radial solution $\psi_1 \in H^1(\mathbb{R}^3)$, then $y_k = 0$ in (4.16). Indeed, if $y_k \rightarrow \infty$, then $\tilde{\varphi}_{\omega_k} \rightharpoonup 0$ weakly. But ψ_1 is a positive radial function. It is impossible. Thus, we have

$$\lim_{\omega \rightarrow \infty} \|\tilde{\varphi}_\omega - \psi_1\|_{H^1} = 0.$$

□

Related to the radial solution ψ_1 , we may define two unbounded self-adjoint operators L_1 and L_2 from L^2 to L^2 by

$$\begin{aligned} L_1 &= -\Delta + 1 - (p-1)(|x|^{-1} * |\psi_1|^p) |\psi_1|^{p-2} - p(|x|^{-1} * (|\psi_1|^{p-1} \cdot)) |\psi_1|^{p-1}, \\ L_2 &= -\Delta + 1 - (|x|^{-1} * |\psi_1|^p) |\psi_1|^{p-2}, \end{aligned}$$

with

$$\begin{aligned} \langle L_1 v, v \rangle &= \|v\|_{H^1}^2 - \mathfrak{R}_{\psi_1, v}(x, y), \\ \langle L_2 v, v \rangle &= \|v\|_{H^1}^2 - \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|\psi_1(x)|^{p-2} |v(x)|^2 |\psi_1(y)|^p}{|x-y|} dx dy, \end{aligned}$$

where

$$\begin{aligned} \mathfrak{R}_{w, v}(x, y) &= (p-1) \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|w(x)|^{p-2} |v(x)|^2 |w(y)|^p}{|x-y|} dx dy \\ &\quad + p \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|w(x)|^{p-2} w(x) |v(x)| |w(y)|^{p-2} w(y) |v(y)|}{|x-y|} dx dy. \end{aligned} \quad (4.17) \quad \boxed{\text{Cross}}$$

We are ready to show some

VOL **Lemma 4.3.** *Let $\eta > 0$ be the constant in Lemma 1.7, there exists $0 < \eta' < \eta$ such that for all p , $2 < p < 2 + \eta'$.*

(1) *There exists $\delta_{01} > 0$ such that*

$$\langle L_1 v, v \rangle \geq \delta_{01} \|v\|_2^2, \quad v \in H_G^1(\mathbb{R}^3, \mathbb{R}),$$

where $(v, \psi_1)_{L^2} = 0$.

(2) *There exists $\delta_{02} > 0$ such that*

$$\langle L_2 v, v \rangle \geq \delta_{02} \|v\|_2^2, \quad v \in H^1(\mathbb{R}^3, \mathbb{R}),$$

where $(v, \psi_1)_{L^2} = 0$.

Proof. (1) By contradiction, suppose that

$$\langle L_1 v_k, v_k \rangle \leq 0, \quad (v_k, \psi_1)_{L^2} = 0, \quad \|v_k\|_{H^1} = 1. \quad (4.18) \quad \boxed{\text{AS}}$$

Since v_k is bounded in $H_G^1(\mathbb{R}^3)$, there exists a subsequence $\{v_i\}$ such that $v_i \rightharpoonup v_0$ weakly in $H_G^1(\mathbb{R}^3)$ and $H_G^1(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$ is compact. Thus

$$(\psi_1, v_0)_{L^2} = 0$$

and

$$\begin{aligned} \lim_{i \rightarrow \infty} \int_{\mathbb{R}^3} (|x|^{-1} * |\psi_1|^p) |\psi_1|^{p-2} v_i^2 &= \int_{\mathbb{R}^3} (|x|^{-1} * |\psi_1|^p) |\psi_1|^{p-2} v_0^2, \\ \lim_{i \rightarrow \infty} \int_{\mathbb{R}^3} (|x|^{-1} * |\psi_1|^{p-1} v_i) |\psi_1|^{p-1} v_i &= \int_{\mathbb{R}^3} (|x|^{-1} * |\psi_1|^{p-1} v_0) |\psi_1|^{p-1} v_0. \end{aligned}$$

By lower semi-continuity, we have

$$\langle L_1 v_0, v_0 \rangle \leq \lim_{i \rightarrow \infty} \langle L_1 v_i, v_i \rangle \leq 0. \quad (4.19) \quad \boxed{\text{OS}}$$

We can prove that $\langle L_1 v_0, v_0 \rangle > 0$. In fact, since

$$S_1 = \inf\{S_0(\varphi); \varphi \in H^1 \setminus \{0\}, I_0(\varphi) = 0\},$$

has a mountain pass characterization with ψ_1 is the mountain pass solution. So the Morse index is at most one. Moreover,

$$\langle L_1 \psi_1, \psi_1 \rangle = \langle L_2 \psi_1, \psi_1 \rangle - (p-1) \int_{\mathbb{R}^3} (|x|^{-1} * |\psi_1|^p) |\psi_1|^p < 0.$$

Thus, L_1 has exactly one negative eigenvalue λ_1 with corresponding eigenfunction e_1 . From Theorem 1.3 of [37], we know there exists $0 < \eta' < \eta$ such that for all p , $2 < p < 2 + \eta'$, the operator L_1 is nondegenerate, that is

$$\Sigma_3 = \text{Ker}\{L_1\} = \text{span}\left\{\frac{\partial \psi_1}{\partial x_j}\right\}, \quad j = 1, 2, 3.$$

Now decomposing $H^1 = \Sigma_1 \oplus \Sigma_2 \oplus \Sigma_3$ with $\Sigma_1 = \text{span}\{e_1\}$, Σ_2 is the image of the spectral projection corresponding to the positive part of the spectrum of L^1 . For $\omega > 0$, set $\psi_{1,\omega} = \omega^{\frac{1}{p-1}} \psi_1(\sqrt{\omega}x)$, then $\psi_{1,\omega}$ satisfies

$$-\Delta \psi_{1,\omega} + \omega \psi_{1,\omega} - (|x|^{-1} * |\psi_{1,\omega}|^p) |\psi_{1,\omega}|^{p-2} \psi_{1,\omega} = 0.$$

Differential the above equation with respect to ω and take $\omega = 1$ to give

$$L_1 \phi = -\Delta \phi + \phi - p(|x|^{-1} * (|\psi_1|^{p-1} \phi)) |\psi_1|^{p-1} - (p-1)(|x|^{-1} * |\psi_1|^p) |\psi_1|^{p-1} \phi = -\psi_1,$$

where $\phi = \frac{\partial \psi_{1,\omega}}{\partial \omega}|_{\omega=1} = \frac{1}{p-1} \psi_1 + \frac{1}{2} x \cdot \nabla \psi_1$. Since ψ_1 is radial, it is easy to check $\phi \in H_G^1$ and $(\frac{\partial \psi_1}{\partial x_j}, \phi)_{L^2} = 0$. We decompose v_0 and ϕ as

$$v_0 = \alpha e_1 + \xi, \quad \phi = \beta e_1 + \eta,$$

where $\alpha, \beta \in \mathbb{R}$ and $\xi, \eta \in \Sigma_2$. If $\alpha = 0$, then $\langle L_1 v_0, v_0 \rangle = \langle L_1 \xi, \xi \rangle > 0$. Suppose $\alpha \neq 0$, then we have

$$\langle L_1 \phi, \phi \rangle = -(\psi_1, \phi) = -(\psi_1, \frac{1}{p-1}\psi_1 + \frac{1}{2}x \cdot \nabla \psi_1) = -\frac{7-3p}{4(p-1)}\|\psi_1\|_2^2 < 0.$$

Therefore, $\beta \neq 0$ and $\langle L_1 \eta, \eta \rangle = \beta^2 \lambda_1 + \langle L_1 \phi, \phi \rangle < \beta^2 \lambda_1$. Furthermore, since $\langle L_1 \phi, v_0 \rangle = -(\psi_1, v_0)_{L^2} = 0 = \alpha \beta \lambda_1 + \langle L_1 \eta, \xi \rangle$. Thus, $\langle L_1 \eta, \xi \rangle = -\alpha \beta \lambda_1$. By Schwarz inequality, we have

$$\langle L_1 v_0, v_0 \rangle = \alpha^2 \lambda_1 + \langle L_1 \xi, \xi \rangle \geq \alpha^2 \lambda_1 + \frac{|\langle L_1 \eta, \xi \rangle|^2}{\langle L_1 \eta, \eta \rangle} > 0,$$

this together with (4.19) lead to $v_0 = 0$. However,

$$\begin{aligned} \lim_{i \rightarrow \infty} \langle L_1 v_i, v_i \rangle &= \lim_{i \rightarrow \infty} (\|v_i\|_{H^1}^2 - \Re_{\psi_1, v_i}(x, y)) \\ &= 1 - \Re_{\psi_1, v_0}(x, y) = 1, \end{aligned}$$

which contradicts with (4.18).

(2) Since $\psi_1(x)$ is the unique positive radial solution of (3.2) with $\mu = 1$. We have

$$L_2 \psi_1 = -\Delta \psi_1 + \psi_1 - (|x|^{-1} * |\psi_1|^p) |\psi_1|^{p-2} \psi_1 = 0,$$

and $\psi_1 > 0$ for $x \in \mathbb{R}^3$, ψ_1 is the first eigenfunction of L^2 corresponding to the eigenvalue 0. Moreover, by Weyl's theorem, the essential spectrum of L^2 are in $[1, \infty)$, since ψ_1 tends to zero at infinity. These conclude (2). \square

For any $v \in X_G$ with $v_1(x) = \operatorname{Re} v(x)$ and $v_2(x) = \operatorname{Im} v(x)$, we introduce two unbounded self-adjoint operators from L^2 to L^2 defined on domain $D(-\Delta + V)$ by

$$\begin{aligned} L_\omega^1 &= -\Delta + \omega + V - (p-1)(|x|^{-1} * |\varphi_\omega|^p) |\varphi_\omega|^{p-2} - p(|x|^{-1} * (|\varphi_\omega|^{p-1} \cdot)) |\varphi_\omega|^{p-1}, \\ L_\omega^2 &= -\Delta + \omega + V - (|x|^{-1} * |\varphi_\omega|^p) |\varphi_\omega|^{p-2}. \end{aligned}$$

Therefore, S_ω'' can be expressed by

$$\langle S_\omega''(\varphi_\omega) v, v \rangle = \langle L_\omega^1 v_1, v_1 \rangle + \langle L_\omega^2 v_2, v_2 \rangle,$$

and

$$\begin{aligned} \langle L_\omega^1 v_1, v_1 \rangle &= \|v_1\|_{X_\omega}^2 - \Re_{\varphi_\omega, v_1}(x, y), \\ \langle L_\omega^2 v_2, v_2 \rangle &= \|v_2\|_{X_\omega}^2 - \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|\varphi_\omega(x)|^{p-2} |v_2(x)|^2 |\varphi_\omega(y)|^p}{|x-y|} dx dy, \\ \operatorname{Re}(\varphi_\omega, v)_{L^2} &= (\varphi_\omega, v_1)_{L^2}, \quad \operatorname{Re}(i\varphi_\omega, v)_{L^2} = (\varphi_\omega, v_2)_{L^2}, \end{aligned} \tag{4.20} \quad \boxed{\text{EQS}}$$

where $\Re_{w,v}(x, y)$ is defined in (4.17).

Let $\lambda_G = \inf\{\langle H v, v \rangle; v \in X_G, \|v\|_{L^2} = 1\}$, for $\omega > -\lambda_G$, we define the rescaled norm $\|\cdot\|_{\tilde{X}_\omega}$ by

$$\|v\|_{\tilde{X}_\omega}^2 = \|v\|_{H^1}^2 + \int_{\mathbb{R}^3} \omega^{-1} V\left(\frac{x}{\sqrt{\omega}}\right) |v(x)|^2 dx, \quad v \in X_G. \tag{4.21} \quad \boxed{\text{RN}}$$

For $v(x) = \omega^{\frac{1}{p-1}} \tilde{v}(\sqrt{\omega}x)$, we define another two re-scaled operators \tilde{L}_ω^1 and \tilde{L}_ω^2 from L^2 to L^2 by

$$\begin{aligned}\tilde{L}_\omega^1 &= -\Delta + 1 + \omega^{-1}V\left(\frac{x}{\sqrt{\omega}}\right) - (p-1)(|x|^{-1} * |\tilde{\varphi}_\omega|^p)|\tilde{\varphi}_\omega|^{p-2} - p(|x|^{-1} * (|\tilde{\varphi}_\omega|^{p-1}))|\tilde{\varphi}_\omega|^{p-1}, \\ \tilde{L}_\omega^2 &= -\Delta + 1 + \omega^{-1}V\left(\frac{x}{\sqrt{\omega}}\right) - (|x|^{-1} * |\tilde{\varphi}_\omega|^p)|\tilde{\varphi}_\omega|^{p-2},\end{aligned}$$

with bilinear form as

$$\begin{aligned}\langle \tilde{L}_\omega^1 v, v \rangle &= \|v\|_{\tilde{X}_\omega}^2 - \Re_{\tilde{\varphi}_\omega, v}(x, y), \\ \langle \tilde{L}_\omega^2 v, v \rangle &= \|v\|_{\tilde{X}_\omega}^2 - \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|\tilde{\varphi}_\omega(x)|^{p-2}|v(x)|^2|\tilde{\varphi}_\omega(y)|^p}{|x-y|} dx dy.\end{aligned}$$

By simple calculation, one can find that

$$\begin{aligned}\|v\|_{\tilde{X}_\omega}^2 &= \omega^{\frac{5-p}{2(p-1)}} \|\tilde{v}\|_{\tilde{X}_\omega}^2, \quad (\varphi_\omega, v)_{L^2} = \omega^{\frac{7-3p}{2(p-1)}} (\tilde{\varphi}_\omega, \tilde{v})_{L^2}, \\ \langle L_\omega^k v, v \rangle &= \omega^{\frac{5-p}{2(p-1)}} \langle \tilde{L}_\omega^k \tilde{v}, \tilde{v} \rangle, \quad k = 1, 2.\end{aligned} \tag{4.22} \quad \boxed{\text{RNO}}$$

VL **Lemma 4.4.** *Assume the conditions (V0) – (V2) hold and $\eta > 0$ be the constant in Lemma 1.7, there exists $0 < \eta' < \eta$, for $2 < p < 2 + \eta'$ and $\varphi_\omega \in \mathcal{M}_\omega$.*

(1) *There exists $\omega_1 > 0$ and $\delta_1 > 0$ such that*

$$\langle L_\omega^1 v, v \rangle \geq \delta_1 \|v\|_{\tilde{X}_\omega}^2, \quad v \in X_G(\mathbb{R}^3, \mathbb{R})$$

for any $\omega \in (\omega_1, \infty)$. Here, $(v, \varphi_\omega)_{L^2} = 0$.

(2) *There exists $\omega_2 > 0$ and $\delta_2 > 0$ such that*

$$\langle L_\omega^2 v, v \rangle \geq \delta_2 \|v\|_{\tilde{X}_\omega}^2, \quad v \in X_G(\mathbb{R}^3, \mathbb{R})$$

for any $\omega \in (\omega_2, \infty)$. Here, $(v, \varphi_\omega)_{L^2} = 0$.

Proof. (1) Arguing by contradiction. If (1) is not true, from (4.22), there exists $\omega_k \rightarrow \infty$ and v_k satisfying $\|v_k\|_{\tilde{X}_{\omega_k}}^2 = 1$ and $(v_k, \tilde{\varphi}_{\omega_k})_{L^2} = 0$ such that

$$\lim_{k \rightarrow \infty} \langle \tilde{L}_{\omega_k}^1 v_k, v_k \rangle \leq 0. \tag{4.23} \quad \boxed{\text{FC}}$$

Since $V_1(x) \geq 0$, from Lemma 2.3 we know

$$\left| \int_{\mathbb{R}^3} \omega_k^{-1} V_2\left(\frac{x}{\sqrt{\omega_k}}\right) |v_k(x)|^2 dx \right| \leq C(\omega_k^{\frac{3}{2q}-1} + \omega_k^{-1}) \|V_2\|_{L^q + L^\infty} \|v_k\|_{H^1}^2.$$

By (4.21), we get

$$\|v_k\|_{H^1}^2 - C(\omega_k^{\frac{3}{2q}-1} + \omega_k^{-1}) \|V_2\|_{L^q + L^\infty} \|v_k\|_{H^1}^2 \leq \|v_k\|_{\tilde{X}_{\omega_k}}^2 = 1, \tag{4.24} \quad \boxed{\text{CR}}$$

thus $\{v_k\}$ is bounded in $H^1(\mathbb{R}^3)$, if ω_k is large enough. Moreover,

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^3} \omega_k^{-1} V_2\left(\frac{x}{\sqrt{\omega_k}}\right) |v_k(x)|^2 dx = 0. \tag{4.25} \quad \boxed{\text{V20}}$$

Let $\{v_i\}$ be a subsequence of $\{v_k\}$ such that $v_i \rightharpoonup v_0$ in $H^1(\mathbb{R}^3)$ and $\{\tilde{\varphi}_{\omega_i}\}$ be of $\{\tilde{\varphi}_{\omega_k}\}$ such that $\tilde{\varphi}_{\omega_i} \rightarrow \psi_1$ in $H^1(\mathbb{R}^3)$, which is due to Lemma 4.2. According to Lemma 2.3 and Hölder inequality, since p is sufficiently close to 2, we also have $|v_i|^2 \rightharpoonup |v_0|^2$ in $L^{3/2}(\mathbb{R}^3)$ and $(|x|^{-1} * |\tilde{\varphi}_{\omega_i}|^p)|\tilde{\varphi}_{\omega_i}|^{p-2} \rightarrow (|x|^{-1} * |\psi_1|^p)|\psi_1|^{p-2}$ in $L^3(\mathbb{R}^3)$ and consequently,

$$\lim_{i \rightarrow \infty} \int_{\mathbb{R}^3} (|x|^{-1} * |\tilde{\varphi}_{\omega_i}|^p)|\tilde{\varphi}_{\omega_i}|^{p-2}|v_i|^2 dx = \int_{\mathbb{R}^3} (|x|^{-1} * |\psi_1|^p)|\psi_1|^{p-2}|v_0|^2 dx. \quad (4.26) \quad \boxed{\text{DC1}}$$

By the analogous analysis, we also have

$$\lim_{i \rightarrow \infty} \int_{\mathbb{R}^3} (|x|^{-1} * |\tilde{\varphi}_{\omega_i}|^{p-2} \tilde{\varphi}_{\omega_i} |v_i|)|\tilde{\varphi}_{\omega_i}|^{p-2} \tilde{\varphi}_{\omega_i} |v_i| dx = \int_{\mathbb{R}^3} (|x|^{-1} * |\psi_1|^{p-2} \psi_1 |v_0|)|\psi_1|^{p-2} \psi_1 |v_0| dx. \quad (4.27) \quad \boxed{\text{DC2}}$$

By (4.23), (4.25), (4.26), (4.27) and $V_1(x) \geq 0$, we have

$$\begin{aligned} 0 &\geq \liminf_{i \rightarrow \infty} \langle \tilde{L}_{\omega_i}^1 v_i, v_i \rangle \\ &= \liminf_{i \rightarrow \infty} \left(\|v_i\|_{H^1}^2 + \int_{\mathbb{R}^3} \omega_i^{-1} V\left(\frac{x}{\sqrt{\omega_i}}\right) |v_i(x)|^2 dx - \mathfrak{R}_{\tilde{\varphi}_{\omega_i}, v_i} \right) \\ &\geq \|v_0\|_{H^1}^2 - \mathfrak{R}_{\psi_1, v_0} = \langle L_1 v_0, v_0 \rangle. \end{aligned}$$

Since $(v_i, \tilde{\varphi}_{\omega_i})_{L^2} = 0$, we have $(v_0, \psi_1)_{L^2} = 0$. While, according to Lemma 4.3 (1), we know $\langle L_1 v_0, v_0 \rangle > 0$, then we can conclude that $v_0 = 0$.

However, by (4.23), (4.26) and (4.27), we have

$$\begin{aligned} 0 &\geq \liminf_{i \rightarrow \infty} \langle \tilde{L}_{\omega_i}^1 v_i, v_i \rangle \\ &= \liminf_{i \rightarrow \infty} \left(\|v_i\|_{\tilde{X}_{\omega_i}}^2 - \mathfrak{R}_{\tilde{\varphi}_{\omega_i}, v_i} \right) \\ &= 1 - \mathfrak{R}_{\psi_1, v_0}, \end{aligned}$$

which means $\mathfrak{R}_{\psi_1, v_0} \geq 1$, this contradicts with the conclusion we just proved that $v_0 = 0$. Hence, (1) is concluded.

Repeat the same arguments, we can prove (2). \square

To show the stability of the standing wave solutions, we need a sufficient condition as follows.

SCD1 Proposition 4.5. *Assume the conditions (V0) – (V2) hold and $\eta > 0$ be the constant in Lemma 1.7, there exists $0 < \eta' < \eta$, for $2 < p < 2 + \eta'$ and $\varphi_\omega \in \mathcal{M}_\omega$, $\omega \in (\omega_0^*, \infty)$ where ω_0^* be the number obtained in Lemma (4.5). There exists $\delta' > 0$ such that*

$$\langle S''_\omega(\varphi_\omega) v, v \rangle \geq \delta' \|v\|_X^2$$

for any $v \in X_G$ satisfying $\text{Re}(\varphi_\omega, v)_{L^2} = 0$ and $\text{Re}(i\varphi_\omega, v)_{L^2} = 0$.

Proof. From (4.20) and according to Lemma 4.4. On the one hand, there exists $\omega_1 > 0$ and $\delta_1 > 0$ such that

$$\langle L_\omega^1 v_1, v_1 \rangle \geq \delta_1 \|v_1\|_{X_\omega}^2, \quad \forall v_1 \in X_G(\mathbb{R}^3, \mathbb{R})$$

for any $\omega \in (\omega_1, \infty)$ and $(v_1, \varphi_\omega)_{L^2} = \operatorname{Re}(\varphi_\omega, v)_{L^2} = 0$. On the other hand, there exists $\omega_2 > 0$ and $\delta_2 > 0$ such that

$$\langle L_\omega^2 v_2, v_2 \rangle \geq \delta_2 \|v_2\|_{X_\omega}^2, \quad \forall v_2 \in X_G(\mathbb{R}^3, \mathbb{R})$$

for any $\omega \in (\omega_2, \infty)$ and $(v_2, \varphi_\omega)_{L^2} = \operatorname{Re}(i\varphi_\omega, v)_{L^2} = 0$. Since there exists $\omega_0 > 0$, \mathcal{M}_ω is not empty. Let $\varphi_\omega(x) \in \mathcal{M}_\omega$, there exists $\omega_0^* = \max\{\omega_1, \omega_2\}$ and $\delta' = \min\{\delta_1, \delta_2\}$ such that

$$\langle S_\omega''(\varphi_\omega)v, v \rangle = \langle L_\omega^1 v_1, v_1 \rangle + \langle L_\omega^2 v_2, v_2 \rangle \geq \delta_1 \|v_1\|_{X_\omega}^2 + \delta_2 \|v_2\|_{X_\omega}^2 \geq \delta' \|v\|_{X_\omega}^2,$$

for $\omega \in (\omega_0^*, \infty)$ and any $v \in X_G$ satisfying $\operatorname{Re}(\varphi_\omega, v)_{L^2} = 0$ and $\operatorname{Re}(i\varphi_\omega, v)_{L^2} = 0$. Then the conclusion follows from the fact that $\|\cdot\|_X$ is equivalent to $\|\cdot\|_{X_\omega}$ on X_G . \square

4.2 Proof of Theorem 1.8

In this subsection, we will show the main result of stability. For any $\varepsilon > 0$ and $\varphi_\omega \in X_G$, we define

$$U_\varepsilon(\varphi_\omega) \triangleq \{v \in X_G; \inf_{\theta \in \mathbb{R}} \|v - e^{i\theta} \varphi_\omega\|_{X_G} < \varepsilon\}.$$

SCD **Lemma 4.6.** *Assume the conditions (V0) – (V2) hold and $\eta > 0$ be the constant in Lemma 1.7, there exists $0 < \eta' < \eta$, for $2 < p < 2 + \eta'$, $\varphi_\omega \in \mathcal{M}_\omega$, $\omega \in (\omega_0^*, \infty)$ where ω_0^* be the number obtained in Lemma (4.5). Then, there exists $C > 0$ and $\varepsilon > 0$ such that*

$$E(u) - E(\varphi_\omega) \geq C \inf_{\theta \in \mathbb{R}} \|u - e^{i\theta} \varphi_\omega\|_X^2,$$

for $u \in U_\varepsilon(\varphi_\omega)$ with $Q(u) = Q(\varphi_\omega)$.

Proof. By the implicit function theorem, if $\varepsilon > 0$ is small enough, $u \in U_\varepsilon(\varphi_\omega)$ with $Q(u) = Q(\varphi_\omega)$, there exists $\theta(u) \in \mathbb{R}$ such that

$$\|u - e^{i\theta(u)} \varphi_\omega\|_X^2 = \min_{\theta \in \mathbb{R}} \|u - e^{i\theta} \varphi_\omega\|_X^2. \quad (4.28) \quad \text{XN}$$

Let $v = e^{-i\theta(u)} u - \varphi_\omega$, we decompose

$$v = a\varphi_\omega + bi\varphi_\omega + y,$$

where $a, b \in \mathbb{R}$, and $y \in X_G$ satisfying $\operatorname{Re}(y, \varphi_\omega)_{L^2} = 0$ and $\operatorname{Re}(y, i\varphi_\omega)_{L^2} = 0$. Obviously,

$$\langle Q'(\varphi_\omega), v \rangle = \operatorname{Re}(\varphi_\omega, v)_{L^2} = \operatorname{Re}(\varphi_\omega, a\varphi_\omega + bi\varphi_\omega + y)_{L^2} = a\|\varphi_\omega\|_2^2.$$

and the Taylor expansion gives

$$Q(\varphi_\omega) = Q(u) = Q(e^{-i\theta(u)} u) = Q(\varphi_\omega + v) = Q(\varphi_\omega) + \langle Q'(\varphi_\omega), v \rangle + O(\|v\|_X^2).$$

Thus, we have

$$a = O(\|v\|_X^2).$$

Moreover, we have

$$S_\omega(u) = S_\omega(e^{-i\theta(u)}u) = S_\omega(\varphi_\omega + v) = S_\omega(\varphi_\omega) + \langle S'(\varphi_\omega), v \rangle + \frac{1}{2} \langle S''_\omega(\varphi_\omega)v, v \rangle + o(\|v\|_X^2),$$

i.e.

$$S_\omega(u) - S_\omega(\varphi_\omega) = \langle S'(\varphi_\omega), v \rangle + \frac{1}{2} \langle S''_\omega(\varphi_\omega)v, v \rangle + o(\|v\|_X^2),$$

by $S'_\omega(\varphi_\omega) = 0$ and $Q(\varphi_\omega) = Q(u)$, we obtain

$$\begin{aligned} E(u) - E(\varphi_\omega) &= S_\omega(u) - \omega Q(u) - (S_\omega(\varphi_\omega) - \omega Q(\varphi_\omega)) \\ &= \frac{1}{2} \langle S''_\omega(\varphi_\omega)v, v \rangle + o(\|v\|_X^2). \end{aligned} \quad (4.29) \quad \boxed{\text{p1}}$$

Next, since $S'_\omega(e^{i\theta}\varphi_\omega) = 0$ for $\theta \in \mathbb{R}$, we have $S''_\omega(\varphi_\omega)i\varphi_\omega = 0$. Therefore,

$$\begin{aligned} \langle S''_\omega(\varphi_\omega)y, y \rangle &= \langle S''_\omega(\varphi_\omega)v, v \rangle - 2a \langle S''_\omega(\varphi_\omega)\varphi_\omega, v \rangle + a^2 \langle S''_\omega(\varphi_\omega)\varphi_\omega, \varphi_\omega \rangle \\ &= \langle S''_\omega(\varphi_\omega)v, v \rangle + O(\|v\|_X^3). \end{aligned} \quad (4.30) \quad \boxed{\text{p2}}$$

Since $y \in X_G$ satisfies $Re(y, \varphi_\omega)_{L^2} = 0$ and $Re(y, i\varphi_\omega)_{L^2} = 0$, by Proposition 4.5, there exists $\delta' > 0$ such that

$$\langle S''_\omega(\varphi_\omega)y, y \rangle \geq \delta' \|y\|_X^2. \quad (4.31) \quad \boxed{\text{p3}}$$

From (4.30) and (4.31), we know

$$\langle S''_\omega(\varphi_\omega)v, v \rangle \geq \delta' \|y\|_X^2 - O(\|v\|_X^3). \quad (4.32) \quad \boxed{\text{p0}}$$

Now, by (4.28) and $(\varphi_\omega, i\varphi_\omega)_X = 0$, we have

$$0 = (v, i\varphi_\omega)_X = b\|\varphi_\omega\|_X^2 + (y, i\varphi_\omega)_X.$$

Thus, we have $|b|\|\varphi_\omega\|_X \leq \|y\|_X$ and $\|v\|_X \leq (|a| + |b|)\|\varphi_\omega\|_X + \|y\|_X \leq 2\|y\|_X + O(\|v\|_X^2)$. Therefore, we have

$$\|y\|_X^2 \geq \frac{1}{4}\|v\|_X^2 + O(\|v\|_X^3). \quad (4.33) \quad \boxed{\text{p4}}$$

By (4.29), (4.32) and (4.33), we have

$$E(u) - E(\varphi_\omega) \geq \frac{\delta'}{2}\|y\|_X^2 + o(\|v\|_X^2) \geq \frac{\delta'}{8}\|v\|_X^2 + o(\|v\|_X^2).$$

Thus, for $u \in U_\varepsilon(\varphi_\omega)$ and $\|v\|_X = \|u - e^{i\theta(u)}\varphi_\omega\|_X < \varepsilon$, we may take $\varepsilon = \varepsilon(\delta') > 0$ small enough to obtain

$$E(u) - E(\varphi_\omega) \geq \frac{\delta'}{8}\|u - e^{i\theta(u)}\varphi_\omega\|_X^2.$$

□

Proof of Theorem 1.8. Argue by contradiction. If the standing wave is unstable, then there exists a sequence of initial data $u_n(0)$ and $\delta > 0$ such that

$$\inf_{\theta \in \mathbb{R}} \|u_n(0) - e^{i\theta} \varphi_\omega\|_X \rightarrow 0,$$

but

$$\sup_{t > 0} \inf_{\theta \in \mathbb{R}} \|u_n(t) - e^{i\theta} \varphi_\omega\|_X \geq \delta,$$

where $u_n(t)$ is a solution with initial value $u_n(0)$. By continuity in t , we can pick the first time t_n so that

$$\inf_{\theta \in \mathbb{R}} \|u_n(t_n) - e^{i\theta} \varphi_\omega\|_X = \delta. \quad (4.34) \quad \boxed{\text{RESULT}}$$

By Proposition 1.2, E and Q are conserved in t . Then, we have

$$E(u_n(t_n)) = E(u_n(0)) \rightarrow E(\varphi_\omega),$$

$$Q(u_n(t_n)) = Q(u_n(0)) \rightarrow Q(\varphi_\omega).$$

There exists a sequence $\{v_n\}$ such that $\|v_n - u_n(t_n)\|_X \rightarrow 0$ and $Q(v_n) = Q(\varphi_\omega)$. Because of the continuity of E , we have $E(v_n) \rightarrow E(\varphi_\omega)$. If we choose δ small enough, from Lemma 4.6, we can obtain

$$c\|v_n - e^{-i\theta(v_n)} \varphi_\omega\|_X^2 = c\|e^{i\theta(v_n)} v_n - \varphi_\omega\|_X^2 \leq E(v_n) - E(\varphi_\omega) \rightarrow 0.$$

Thus, $\|u_n(t_n) - e^{-i\theta(v_n)} \varphi_\omega\|_X \rightarrow 0$, which contradicts (4.34).

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